Name: Solution Key

Directions: Please be sure to answer each question carefully in the space provided. Please present your work neatly so you receive maximum credit. If you have any questions please feel free to ask.

Midterm Exam: November 17th, 2005
Professor: Paul J. Atzberger

Scoring:

Problem 1: 10

Problem 2: 20

Problem 3: 20

Problem 4: 10 + bonus 10

Final Score: 60
Problem 1: Let $X_1$ and $X_2$ denote the following independent random variables:

\[ X_1 = \begin{cases} 
0 & \text{probability of } 1/2 \\
1 & \text{probability of } 1/2 
\end{cases} \]

and

\[ X_2 = \begin{cases} 
-1 & \text{probability of } 1/4 \\
1 & \text{probability of } 3/4 
\end{cases} \]

Let $Y = X_1 + X_2$ and $Z = X_1 X_2$.

1) Compute the mean of $Y$ and the mean of $Z$.

\[
\begin{align*}
\mu_Y &= E[Y] = E[X_1 + X_2] = E[X_1] + E[X_2] \quad \text{(by linearity of } E[\cdot] \text{)} \\
&= (0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}) + (-1 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4}) \\
&= \frac{1}{2} + \frac{3}{4} = 1. \\
\mu_Z &= E[Z] = E[X_1 X_2] = E[X_1] E[X_2] \quad \text{(by independence of } X_1, X_2 \text{)} \\
&= (0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}) \cdot (-1 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4}) \\
&= \frac{1}{4}.
\end{align*}
\]

2) Compute the variance of $Y$ and the variance of $Z$.

\[
\begin{align*}
\sigma_Y^2 &= E[(Y - \mu_Y)^2] = E[Y^2] - \mu_Y^2 = E[(X_1 + X_2)^2] - \mu_Y^2 \\
&= E[X_1^2] + 2E[X_1 X_2] + E[X_2^2] - \mu_Y^2 \\
&= (0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2}) + 2 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{3}{4} - 1^2 \\
&= \frac{1}{2} + \frac{1}{2} + 1 - 1 = 1. \\
\sigma_Z^2 &= E[(Z - \mu_Z)^2] = E[(X_1 X_2)^2] - \mu_Z^2 = E[X_1^2] E[X_2^2] - \mu_Z^2 \\
&= \sigma_Y^2 = \frac{1}{2}. \\
\end{align*}
\]

3) Compute the covariance of $Y$ and $Z$.

\[
\begin{align*}
\sigma_{YZ} &= E[(Y - \mu_Y)(Z - \mu_Z)] = E[YZ] - \mu_Y \mu_Z \\
&= E[(X_1 + X_2)(X_1 X_2)] - 1 \cdot \frac{1}{2} = E[X_1 X_2] + E[X_2^2 X_1] \\
&= E[X_1^2] E[X_2] + E[X_2^2] E[X_1] \\
&= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} - \frac{1}{4} \\
&= \frac{1}{4}.
\end{align*}
\]
Problem 2: Let us consider the multiperiod binomial market model given by:

![Tree Diagram]

Figure 1:

Let the interest rate \( r = \ln(1.1/10) \) and the time elapsed between the periods be one year, \( \delta t = 1 \). We shall consider two options having payoff functions \( f(s) = (s - K)^3 \) and \( g(s) = (K - s)^2 \) with strike price \( K = 6 \).

1) Compute the risk-neutral probabilities associated with each branch of the tree. (Hint: Consider each node of the tree and the probability required for an up or down step to be consistent with no-arbitrage when pricing an option by the risk-neutral expected payoff.)

\[
q = \frac{e^{r\delta t} S_{\text{up}} - S_{\text{down}}}{S_{\text{up}} - S_{\text{down}}}
\]
2) Compute at time $t = 1$ for the spot price $S(t) = 4$ the value of the option having payoff $f(s)$ at maturity by using the risk-neutral probabilities.

\[
\begin{align*}
    f(0) &= (0 - 6)^+ = 0 \\
    f(4) &= (4 - 6)^+ = 0 \\
    f(8) &= (8 - 6)^+ = 8
\end{align*}
\]

\[
V(f_j T - \delta t, 2) = e^{-r \delta t} \left( q f(4) + (1-q)f(0) \right) = 0
\]

\[
= \frac{10}{11} \left( 0 \cdot 0.55 + 0 \cdot 0.45 \right) = 0
\]

\[
V(f_j T - \delta t, 6) = \frac{10}{11} \left( 0.65 \cdot 8 + 0.35 \cdot 0 \right) = \frac{52}{11} = 4.7273
\]

\[
V(f_j T - 2\delta t, 4) = \frac{10}{11} \left( 0.6 \cdot 4.7273 + 0.4 \cdot 0 \right) = \frac{51.2}{11} = 4.655
\]

3) Compute at time $t = 1$ the the spot price $S(t) = 4$ the value of the option having payoff $g(s)$ at maturity by using the risk-neutral probabilities.

\[
\begin{align*}
    g(0) &= 36 \\
    g(4) &= 4 \\
    g(8) &= 0 \\
    g(10) &= 0
\end{align*}
\]

\[
V(f_j T - \delta t, 2) = \frac{10}{11} \left( 0.55 \cdot 4 + 0.45 \cdot 36 \right)
\]

\[
= \frac{10}{11} \left( 22 + 16.2 \right) = \frac{184}{11} = 16.7273
\]

\[
V(f_j T - \delta t, 6) = \frac{10}{11} \left( 0.35 \cdot 4 \right) = \frac{14}{11} = 1.2727
\]

\[
V(f_j T - 2\delta t, 4) = \frac{10}{11} \left( 0.6 \cdot \frac{14}{11} + 0.4 \cdot \frac{184}{11} \right)
\]

\[
= \frac{1}{11} \left( 8.4 + 73.6 \right) = \frac{82}{11} = 7.5636
\]
4) At time $t = 1$ and spot price $S(t) = 4$, what is the value of an option having the payoff $h(s) = f(s) + 2g(s)$?

$$V(\tilde{f} + 2g; T - 2\delta t, 4) = V(\tilde{f}; T - 2\delta t, 4) + 2V(g; T - 2\delta t, 4)$$

$$= \frac{\tilde{v}1}{1 + \tilde{r}} + 2 \cdot \frac{\tilde{v}0}{1 + \tilde{r}} = \frac{10.52}{1 + \tilde{r}} = 16.15$$

5) For the option with payoff $f(s)$ at maturity, compute the replicating portfolio for the single period following when the spot price is $S(t) = 9$. Denote the stock position of the replicating portfolio by $\phi$ and the bond position by $\psi$.

$f(10) = 64$

$f(1) = 8$

$$V(t_0) = \phi S(t_0) + \psi e^{-r\delta t}$$

$$V(T) = \phi S(T) + \psi = f(S(T))$$

For the binomial market this gives the two conditions

$$\begin{align*}
\phi \cdot 10 + \psi &= 64 \\
\phi \cdot 8 + \psi &= 8
\end{align*}$$

$$\begin{align*}
\phi &= 5.6 \\
\phi &= 28
\end{align*}$$

$$\psi = 8 - 28 \cdot 8 = 8 - 224 = -216$$

$$\boxed{\begin{align*}
\phi &= 28 \\
\psi &= -216
\end{align*}}$$
Problem 3: Consider a market consisting of 3 risky assets having the covariance matrix:

\[
V = \begin{bmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 6 \\
\end{bmatrix}
\]

and the expected returns:

\[
\mu = \begin{bmatrix}
1 \\
3 \\
4 \\
\end{bmatrix}
\]

1) Find the weights \( w_1, w_2, w_3 \) of a portfolio having an expected return \( \mu_p = 3/2 \) with the minimum variance. In other words, find the portfolio with weights satisfying Markowitz's optimization problem:

\[
\min \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j V_{i,j} \\
\text{subject:} \\
\sum_{i=1}^{3} w_i - 1 = 0 \\
\sum_{i=1}^{3} w_i \mu_i - \mu_p = 0.
\]

(Hint: Use the Method of Lagrange Multipliers)

\[\begin{align*}
L (w_1, w_2, w_3, \lambda_1, \lambda_2, \lambda_3) &= \frac{1}{2} \left( w_1^2 + 4 w_2^2 + 6 w_3^2 \right) \\
&\quad - \lambda_1 \left( w_1 + w_2 + w_3 - 1 \right) \\
&\quad - \lambda_2 \left( w_1 + 3 w_2 + 4 w_3 - 3/\mu_p \right)
\end{align*}\]

\[
\begin{cases}
\frac{\partial L}{\partial w_1} = w_1 - \lambda_1 - \lambda_2 = 0 \\
\frac{\partial L}{\partial w_2} = 4 w_2 - \lambda_1 - 3 \lambda_2 = 0 \\
\frac{\partial L}{\partial w_3} = 6 w_3 - \lambda_1 - 4 \lambda_2 = 0
\end{cases}
\]

Solving \((4)\) in terms of \( \lambda_1, \lambda_2 \) we obtain:

\[
w_1 = \lambda_1 + \lambda_2, \quad w_2 = \frac{\lambda_1 + 3 \lambda_2}{4}, \quad w_3 = \frac{\lambda_1 + 4 \lambda_2}{6}
\]

We now solve for \( \lambda_1, \lambda_2 \):

\[
\begin{cases}
-(\lambda_1 + \lambda_2 + \frac{\lambda_1 + 3 \lambda_2}{4} + \frac{\lambda_1 + 4 \lambda_2}{6} - 1) = 0 \\
-(\lambda_1 + \lambda_2 + \frac{3 \lambda_1 + 4 \lambda_2}{4} + \frac{4 \lambda_1 + 6 \lambda_2 - 3/\mu_p}{6}) = 0
\end{cases}
\]

\[
\begin{cases}
2 \lambda_1 + 2 \lambda_2 + 6 \lambda_1 + 18 \lambda_2 + 4 \lambda_1 + 16 \lambda_2 = 24 \\
2 \lambda_1 + 2 \lambda_2 + 18 \lambda_2 + 54 \lambda_2 + 16 \lambda_1 + 64 \lambda_2 = 36
\end{cases}
\]
\[
\begin{align*}
34 \lambda_1 + 58 \lambda_2 &= 24 \\
58 \lambda_1 + 142 \lambda_2 &= 36 \\
17 \lambda_1 + 29 \lambda_2 &= 12 \\
29 \lambda_1 + 71 \lambda_2 &= 18 \\
29 \lambda_1 + 12 \lambda_2 &= 18 \\
\end{align*}
\]

\[
\left(71 - \frac{(29)^2}{17}\right) \lambda_2 = \left(\frac{29}{17}, 12 + 18\right)
\]

\[
\lambda_2 = \frac{\left(18 - \frac{29}{17}, 12\right)}{\left(71 - \frac{(29)^2}{17}\right)} = -0.1148
\]

\[
\lambda_1 = \frac{12 - 29 \lambda_2}{17} = 0.9016
\]

Substituting for \(\lambda_1, \lambda_2\) in the expressions for \(w_1, w_2, w_3\) gives the portfolio weights:

\[
w_1 = 0.7869
\]

\[
w_2 = 0.6475
\]

\[
w_3 = 0.5810
\]

This was obtained from above, which is equivalent to 
\(w = V^{-1}(\lambda_1 \bar{z}^2 + \lambda_2 \bar{m})\).
Problem 4: The Black-Scholes Formulas for the call and put option are given by:

\[
\begin{align*}
c(s_0, T; K) &= s_0 N(d_1) - Ke^{-rT} N(d_2) \\
p(s_0, T; K) &= Ke^{-rT} N(-d_2) - s_0 N(-d_1)
\end{align*}
\]

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left( \ln\left(\frac{s_0}{K}\right) + (r + \frac{1}{2} \sigma^2)T \right)
\]

\[
d_2 = \frac{1}{\sigma \sqrt{T}} \left( \ln\left(\frac{s_0}{K}\right) + (r - \frac{1}{2} \sigma^2)T \right)
\]

\[
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy.
\]

1) Does the value of the call option increase or decrease when the spot price of the underlying stock increases?

*Increases. Intuitively one expects this since having a greater spot price suggests it is more likely for the stock to end up at maturity in the money.*

2) Does the value of the put option increase or decrease when the volatility of the underlying stock increases?

*Increases. Again, one expects this since the stock price is now less certain so there is a greater chance of the stock ending in the money.*

**Bonus:**

3) Compute the Delta, \( \Delta = \frac{\partial c}{\partial s_0} \), of the call option using the Black-Scholes Formula. (Hint: use that \( d_2 = d_1 - \sigma \sqrt{T} \) to simplify your final expression.)

\[
\frac{\partial c}{\partial s_0} = N(d_1) + s_0 N'(d_1) \frac{\partial d_1}{\partial s_0} - Ke^{-rT} N'(d_2) \frac{\partial d_2}{\partial s_0}
\]

\[
N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}, \quad \frac{\partial d_1}{\partial s_0} = \frac{\partial d_2}{\partial s_0} = \frac{1}{\sqrt{2\pi} s_0} \frac{1}{\sigma \sqrt{T}}
\]

(4) \( d_2 = d_1 - \sigma \sqrt{T} d_1 + \sigma \sqrt{T} \)

\[
d_2 = d_1 - 2 \ln\left(\frac{s_0}{K}\right) - rT
\]

\[
\frac{\partial c}{\partial s_0} = N(d_1) + \frac{s_0}{\sqrt{2\pi}} \left( e^{-\frac{d_1^2}{2}} - \frac{K}{s_0} e^{-rT} e^{-\frac{d_2^2}{2}} \right)
\]

\[
= N(d_1) \quad \text{by (4)}.
\]

This shows \( \frac{\partial c}{\partial s_0} > 0 \) justifying 4.1.
4) Compute the Vega, \( v = \frac{\partial c}{\partial \sigma} \), of the call option using the Black-Scholes Formula. (Hint: use that \( d_2 = d_1 - \sigma \sqrt{T} \) to simplify your final expression.)

\[
\frac{\partial c}{\partial \sigma} = S_0 N'(d_1) \frac{\partial d_1}{\partial \sigma} - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial \sigma}
\]

\( N'(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}} \)

\( \frac{\partial d_1}{\partial \sigma} = -\frac{1}{\sigma} d_1 + \sqrt{T} \)

\( \frac{\partial d_2}{\partial \sigma} = -\frac{1}{\sigma} d_2 - \sigma \sqrt{T} = -\frac{1}{\sigma}(d_1 - \sigma \sqrt{T}) - \sigma \sqrt{T} = -\frac{1}{\sigma} d_1 \)

\((\star)\) \( d_2^* = d_1^* - 2 \ln \left( \frac{S_0}{K} \right) - 2rT \)

\[
\frac{\partial c}{\partial \sigma} = S_0 \left( -\frac{1}{\sigma} d_1 + \sqrt{T} \right) e^{\frac{-d_1^*}{2}} - \frac{K e^{-rT}}{\sqrt{2\pi \sigma^2}} \left( -\frac{1}{\sigma} d_1 S_0 \right) e^{\frac{-d_2^*}{2}}
\]

\[
= S_0 \frac{\sqrt{T}}{\sqrt{2\pi \sigma^2}} e^{\frac{-d_1^*}{2}} - \frac{1}{\sigma} d_1 S_0 \left( e^{\frac{-d_2^*}{2}} - \frac{K e^{-rT}}{S_0} e^{\frac{-d_2^*}{2}} \right)
\]

\[
= S_0 \frac{\sqrt{T}}{\sqrt{2\pi \sigma^2}} e^{\frac{-d_1^*}{2}} \text{ by } (\star) \text{.}
\]

This shows \( \frac{\partial c}{\partial \sigma} > 0 \), justifying 4.2.