Midterm Practice Exam:  
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Introduction to Numerical Analysis, 104A  
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**Directions:** Answer each question carefully and be sure to show all of your work. You are permitted to use a calculator but please be sure to show intermediate steps in your calculations. If you have any questions please feel free to ask.
Problem 1: Compute the absolute and relative errors when approximating $p$ by the value $p^*$.

$$
\text{Err} = |p^* - p|, \quad \text{Err}_{rel} = \frac{|p^* - p|}{|p|}
$$

a) $p = \pi$ by $p^* = 3.14$

$$
\begin{align*}
\text{Err}_{abs} &= 1.59265 \times 10^{-3} \\
\text{Err}_{rel} &= 5.0696 \times 10^{-4}
\end{align*}
$$

b) $p = e^1$ by $p^* = 1699/625$

$$
\begin{align*}
\text{Err}_{abs} &= 1.18172 \times 10^{-4} \\
\text{Err}_{rel} &= 4.84729 \times 10^{-5}
\end{align*}
$$

c) $p = 6!$ by $p^* = 359$

$$
\begin{align*}
\text{Err}_{abs} &= 361 \\
\text{Err}_{rel} &= 5.01389 \times 10^{-1}
\end{align*}
$$

d) $p = \cos(0.1)$ by $p^* = 1 - (0.1^2/2)$

$$
\begin{align*}
\text{Err}_{abs} &= 4.16528 \times 10^{-6} \\
\text{Err}_{rel} &= 4.18619 \times 10^{-6}
\end{align*}
$$
Problem 2: In this problem we shall use k-digit-chopping to model the role of floating point arithmetic. In particular, you shall compare the accuracy when using different formulas to compute an approximate $p^*$ to $p$. For each problem below do the following: (i) compute the relative errors using each formula, (ii) state which of the two formulas $p^*_1$ or $p^*_2$ gives a more accurate approximation.

a) For the 4-digit-chopping model for the number representation and arithmetic use each of the formulas to compute the approximate $p^*$ of $p$:

Formula 1: $p^*_1 = \sqrt{2} \cdot \sqrt{3} + \sqrt{3} \cdot \pi + \sqrt{2} \cdot \pi + 2$

Formula 2: $p^*_2 = (\sqrt{2} + \pi) \cdot (\sqrt{3} + \sqrt{2})$

The near-exact solution is given by $p = 14.33377077364420$.

b) For the 5-digit-chopping model for the number representation and arithmetic use each of the formulas to compute the approximate $p^*$ of $p$:

Formula 1: $p^*_1 = (\sqrt{2.1} - \sqrt{2})/0.1$

Formula 2: $p^*_2 = (\sqrt{2.01} - \sqrt{2})/0.01$

The near-exact solution is given by $p = 0.35355339059327$. 
Problem 2: (k-digit chopping, \(k = 4\))

\[
\sqrt{2} \approx 1.414 \\
\sqrt{3} \approx 1.732 \\
\pi \approx 3.141
\]

**Formula 1:**
\[
\sqrt{2} \cdot \sqrt{3} \approx 2.449 \\
\sqrt{3} \cdot \pi \approx 5.440 \\
\sqrt{5} \cdot \pi \approx 4.441
\]

\[
(\sqrt{2} \cdot \sqrt{3}) + (\sqrt{3} \cdot \pi) \approx 7.889 \\
((\sqrt{2} \cdot \sqrt{3}) + (\sqrt{3} \cdot \pi)) + (\sqrt{5} \cdot \pi) \approx 12.882 \\
((\sqrt{2} \cdot \sqrt{3}) + (\sqrt{3} \cdot \pi)) + 2 \pi \approx 14.333 : = \rho_1^*
\]

**Formula 2:**
\[
(\sqrt{2} + \pi) \approx 4.555 \\
(\sqrt{3} + \sqrt{5}) \approx 3.466 \\
(\sqrt{2} + \pi) \cdot (\sqrt{3} + \sqrt{5}) \approx 14.333 : = \rho_2^*
\]

In each case
\[
\varepsilon_{rel,1} = \frac{|\rho_1^* - \rho_1|}{|\rho_1|} \approx 2.6 \times 10^{-4}
\]
\[
\varepsilon_{rel,2} = \frac{|\rho_2^* - \rho_2|}{|\rho_2|} \approx 2.6 \times 10^{-4}
\]

For this problem we find that the relative errors are equal, so far accuracy either formula 1 or formula 2 could be used. However, formula 1 requires fewer arithmetic operations and generally that approach will have better accuracy, so formula 1 is preferred.
b) (k-digit chopping, k = 5)

\[ \sqrt{11} \approx 1.4491 \]
\[ \sqrt{2} \approx 1.4142 \]

\[ \sqrt{11} - \sqrt{2} \approx 3.49 \times 10^{-2} \]
\[ (\sqrt{11} - \sqrt{2})/0.1 \approx 3.49 \times 10^{-1} : = \rho^* \]

\[ \sqrt{2.01} \approx 1.4177 \]
\[ \sqrt{2} \approx 1.4142 \]

\[ (\sqrt{2.01} - \sqrt{2})/0.01 \approx 3.5 \times 10^{-1} : = \rho^* \]

The relative errors are

\[ \varepsilon_{rel,1} = \frac{|\rho^* - p|}{p} = 1.288 \times 10^{-2} \]

\[ \varepsilon_{rel,2} = \frac{|\rho^* - p|}{p} = 1.005 \times 10^{-2} \]

For this problem we find that the second formula performed better. In fact \( p = \frac{5}{\sqrt{x}} \) and above is a finite difference approx. to \( \frac{d}{dx} \sqrt{x} \) evaluated at \( x = 2 \).
Problem 3: Compute the relative errors of when approximating $p$ by the value $p^*$ and state the number of significant digits.

\[ \epsilon_{rel} = \frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t} \text{ then } t \text{ significant digits.} \]

a) $p = \pi$ by $p^* = \frac{22}{7}$

\[ \epsilon_{rel} = 4.025 \times 10^{-4} \leq 5 \times 10^{-4}, \text{ } t = 4, \text{ significant digits.} \]

b) $p = e^2$ by $p^* = \frac{2886601}{390625}$

\[ \epsilon_{rel} = 8.695 \times 10^{-5} \leq 5 \times 10^{-4}, \text{ } t = 4. \]

c) $p = 5!$ by $p^* = 121$

\[ \epsilon_{rel} = 8.3 \times 10^{-5} \leq 5 \times 10^{-4}, \text{ } t = 2. \]

d) $p = \sin(0.1)$ by $p^* = 0.1 - (0.1^3/6)$

\[ \epsilon_{rel} = 8.3453 \times 10^{-7} \leq 5 \times 10^{-6}, \text{ } t = 6. \]
Problem 4: Give the rate of convergence of $f(n) \to 0$ as $n \to \infty$ of the following expressions. State your final result using $\text{"Big Oh"}$ notation in the form $f(n) = O(g(n))$, where $g(n) = n^{-p}$ for some $p$.

a) $f(n) = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \leq \frac{2}{n}$

$\therefore f(n) = O\left(\frac{1}{n}\right)$

b) $|f(n)| = \left|\frac{\sin(n)+1}{n^2}\right| \leq \frac{|\sin(n)|+1}{n^2} \leq \frac{2}{n^2}$

$\therefore f(n) = O\left(\frac{1}{n^2}\right)$

c) $f(n) = \frac{e^{1/n}+n}{n^3} \leq \frac{e+n}{n^3} = \frac{e}{n^3} + \frac{1}{n^2} \leq \frac{e+1}{n^2}$

$\therefore f(n) = O\left(\frac{1}{n^2}\right)$

d) $f(n) = \frac{n+1}{(n-1)(n+1)} = \frac{\frac{n+1}{n}}{(n-1)(n+1)} = \frac{1}{n-1} \leq \frac{1}{n} \left(1+\frac{1}{n} \right)$

$\leq \frac{2}{n}, \quad (n \geq 2)$

$\therefore f(n) = O\left(\frac{1}{n}\right)$,
Problem 5: Finding Roots and Fixed Points

a) If we are asked to find the root of \( f(x) = x^2 - 1 \) in \([0, 2]\), give a function \( g(x) \) for which the root is a fixed point.

\[ g(x) = x^2 + x - 1 \]

b) State the criteria for the function \( g(x) \) to have a fixed point in the interval \([a, b]\). Does the function \( g(x) = \sqrt{x} \) have a fixed point in the interval \([0.5, 2]\)?

A function \( g(x) \) has a fixed point in \([a, b]\), if

(i) \( g(x) \) is continuous,

(ii) \( g(x) \in [a, b] \) for \( x \in [a, b] \).

For \( g(x) = \sqrt{x} \), \( g \) is continuous and \( g(0.5) = g(1) = 1.1181 \), so \( g(x) \notin [0.5, 1] \) and does not have any fixed points in this interval.

c) State the criteria for a function \( g(x) \) used in a fixed point iteration method to converge on an interval \([a, b]\). Does the function \( g(x) = 0.5 \times x + \frac{1}{x} \) meet this criteria on the interval \([1, 2]\)?

A function \( g(x) \) converges to a fixed point if in addition to (i), (ii) above, we have that \( |g'(x)| < 1 \) for \( x \in [a, b] \). \( |g'(x)| = \frac{1}{2} + \frac{1}{x} \) does meet the criteria.

d) If \( p_n \) is the \( n^{th} \) approximate of the root \( p \) in a root finding algorithm, the error is given by \( e_n = |p_n - p| \) and the error for the next iteration is given by \( e_{n+1} = |p_{n+1} - p| \). The rate of convergence is defined as \( m \) if

\[ \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^m} \leq C \]

for some finite constant \( C \) (technically this should be \( \lim \sup \)).

State the rate of convergence of the Bisection Method and Newton’s Method (give \( m \)). Suppose we are very close to the root \( p \), in the sense that the left and right hands sides in the limit above can be approximately equated. For an error of \( e_n = 10^{-2} \) give an estimate of the error \( e_{n+1} \) in the case \( C = 1 \) when using the Bisection Method. Give the estimate when using Newton’s Method.

**Bisection Method:** \( e_{n+1} \approx e_n = 10^{-2} \)

**Newton Method:** \( e_{n+1} \approx e_n = 10^{-4} \)

For Newton, a very drastic improvement in accuracy is obtained over one iteration. For Bisection, the typical improvement is at most \( \frac{1}{2} \) at each step.