Name: Solution Key

Midterm Exam:
Professor: Paul J. Atzberger
Introduction to Numerical Analysis, 104A
November 7th, 2006

Scoring:
Problem1: ____________
Problem2: ____________
Problem3: ____________
Problem4: ____________

Directions: Answer each question carefully and be sure to show all of your work. You are permitted to use a calculator but please be sure to show intermediate steps in your calculations. If you have any questions please feel free to ask.
**Problem 1**: Compute the absolute and relative errors when approximating \( p \) by the value \( p^* \).

\[ p = \pi \text{ by } p^* = 3.1416 \]

\[ \varepsilon_{\text{abs}} = 7.3464 \times 10^{-6} \]
\[ \varepsilon_{\text{rel}} = 2.3584 \times 10^{-6} \]

b) \( p = e^{\pi} \text{ by } p^* = 1157/50 \)

\[ \varepsilon_{\text{abs}} = 6.9363 \times 10^{-4} \]
\[ \varepsilon_{\text{rel}} = 2.9331 \times 10^{-5} \]
c) \( p = \frac{1}{1-0.1} \) by \( p^* = 1 + 0.1 \)
\[ \xi_{\text{abs}} = 1.1 \times 10^{-5} \]
\[ \xi_{\text{rel}} = 1.01 \times 10^{-5} \]

d) \( p = 10! \) by \( p^* = \sqrt{2\pi 10} e^{-10}(10^{10}) \)
\[ \xi_{\text{abs}} = 3.0 \times 10^4 \times 10^4 \]
\[ \xi_{\text{rel}} = 8.196 \times 10^{-3} \]
Problem 2: In this problem we shall use $k$-digit-chopping to model the role of floating point arithmetic. In particular, you shall compare the accuracy when using different formulas to compute an approximate $p^*$ to $p$. For each problem below do the following: (i) compute the relative errors using each formula, (ii) state which of the two formulas $p_1^*$ or $p_2^*$ gives a more accurate approximation.

a) For the 3-digit-chopping model for the number representation and arithmetic use each of the formulas to compute the approximate $p^*$ of $p$:

**Formula 1:** $p_1^* = (e^1 \cdot \pi + \sqrt{2} \cdot \pi + e^1 \sqrt{8}) + 4$

**Formula 2:** $p_2^* = (\sqrt{8} + \pi) \cdot (\sqrt{2} + e^1)$

The near-exact solution is given by $p = 24.67107921715017$.

---

**Formula 1:**

\[
\begin{align*}
\text{e}^1 & \approx 2.71 \\
\pi & \approx 3.14 \\
\sqrt{2} & \approx 1.41 \\
\sqrt{8} & \approx 2.82 \\
\end{align*}
\]

\[
\begin{align*}
e^1 \cdot \pi & \approx 8.50 \\
\sqrt{2} \cdot \pi & \approx 4.42 \\
e^1 \cdot \sqrt{8} & \approx 7.64 \\
\end{align*}
\]

\[
\begin{align*}
e^1 \cdot \pi + \sqrt{2} \cdot \pi & \approx 12.9 \\
e^1 \cdot \pi + \sqrt{2} \cdot \pi + e^1 \cdot \sqrt{8} & \approx 20.5 \\
e^1 \cdot \pi + \sqrt{2} \cdot \pi + e^1 \cdot \sqrt{8} + 4 & \approx 24.5 \\
\end{align*}
\]

\[\rho_1^* = p^*\]

**Formula 2:**

\[
\begin{align*}
\sqrt{8} + \pi & \approx 5.96 \\
\sqrt{8} + e^1 & \approx 4.13 \\
(\sqrt{8} + \pi) \cdot (\sqrt{8} + e^1) & \approx 24.5 \\
\end{align*}
\]

\[\rho_2^* = p^*\]

\[\varepsilon_{1, rel} = 6.934 \times 10^{-3}\]

\[\varepsilon_{2, rel} = 6.934 \times 10^{-3}\]

Both approximations equally well $p$. 

b) For the 2-digit-chopping model for the number representation and arithmetic use each of the formulas to compute the approximate $p^*$ of $p$:

*Formula 1:* $p^*_1 = 1 - (0.1)^2$

*Formula 2:* $p^*_2 = 1 - \frac{(0.1)^2}{2}$

The near-exact solution is given by $p = 0.99500416527803$.

*Formula 1:
$(0.1)(0.1) \approx 0.010$
$1 - (0.1)(0.1) \approx 0.99 \approx p^*_1$

*Formula 2:
$(0.1)(0.1) \approx 0.010$
$\frac{(0.1)^2}{2} \approx 0.0050$
$1 - \frac{(0.1)(0.1)}{2} \approx 1.0 - 0.0050 \approx 0.9950 \approx p^*_2$

$\varepsilon_{1, \text{rel}} = 5.029 \times 10^{-3}$
$\varepsilon_{2, \text{rel}} = 5.029 \times 10^{-3}$

In this case approximate formulas are equally accurate.
Problem 3: Compute the relative errors of when approximating $p$ by the value $p^*$ and state the number of significant digits.

a) $p = \pi$ by $p^* = 111/35$

\[ \varepsilon_{rel} = 4.971 \times 10^{-5} \leq 5 \times 10^{-3}, \quad t = 2 \]

b) $p = e^2$ by $p^* = (1 - \frac{2}{10})^{10}$

\[ \varepsilon_{rel} = 0.8547 \times 10^{-1} \leq 5 \times 10^{-2}, \quad t = 0 \]
c) $p = 5!$ by $p^* = \sqrt{2\pi}5e^{-5^2}5^5$

$$\varepsilon_{rel} = 1.65051 \times 10^{-3}, \quad t = 2,$$

---

d) $p = e^{0.1}$ by $p^* = 1 + 0.1 + (0.1^2/2) + (0.1^3/6)$

$$\varepsilon_{rel} = 3.8468 \times 10^{-6} \leq 5 \times 10^{-6}, \quad t = 6$$
Problem 4: Give the rate of convergence of $f(n) \to 0$ as $n \to \infty$ of the following expressions. State your final result using "Big Oh" notation in the form $f(n) = O(g(n))$, where $g(n) = n^{-p}$ for some $p$.

a) $f(n) = \frac{n^2+1}{n^3} \leq \frac{3n^3}{n^3} = \frac{3}{n} \implies f(n) = O\left(\frac{1}{n}\right)$

b) $f(n) = \frac{n^3+2n^2+1}{n^2+1} \leq \frac{3n^3}{n^3} = \frac{3n^3}{n^3} = \frac{3n^3}{n^3} \implies f(n) = O(n)$
c) \( f(n) = \frac{e^{1/n} + n^2}{n^3} \leq \frac{3 + n^3}{n^4} \leq \frac{4n^3}{n^4} = \frac{4}{n} \). \( f(n) = O\left(\frac{1}{n}\right) \).

d) \( f(n) = \frac{n^2 - 1}{(n-1)(n+1)n} = \frac{(n-1)(n+1)}{(n-1)(n+1)n} = \frac{1}{n} \). \( f(n) = O\left(\frac{1}{n}\right) \).