Take Home Final

Introduction to Complex Variable
Professor: Paul J. Atzberger
Due: Monday, December 17, 5:00pm

Instructions: In the take home final you will use the materials discussed in class to solve each of the problems. You should write up your solutions clearly and concisely similarly to the in-class assignments. You should work independently on your solutions. If you have questions please feel free to come to office hours or schedule an appointment for assistance.

Part 1: Steady-State Heat Equations
The steady-state temperature distribution \( T(x, y) \) in a 2D body \( \Omega \) is given by
\[
T_{xx} + T_{yy} = 0, z \in \Omega
\]
\[
T(x, y) = \phi(x, y), z \in \partial \Omega
\]
where \( \phi(x, y) \) is a given temperature distribution on the boundary \( \partial \Omega \) of the body.

a) Show that the real part of the function \( f(z) = \sin(z) \) solves the steady-state heat equation on \( \Omega_1 \). In particular, show that the proposed solution satisfies the boundary conditions and is harmonic, see Figure 1 for the region \( \Omega_1 \) and the boundary conditions.

b) Show that the function \( h(z) = e^z \) maps the domain \( \Omega_1 \) to \( \Omega_2 \). Be sure to show with orientation how the trace around the boundary of the domain \( \Omega_1 \) maps to the boundary of \( \Omega_2 \). Give the images on the boundary of \( \Omega_2 \) of the points \( A, B, C, D \) labeled on \( \Omega_1 \).

c) Let \( g(w) = \log(w) \) be the inverse of \( h(z) \) with \( \log(w) = \ln(|w|) + i\arg(w) \), where \(-\pi/2 < \arg(w) \leq 3\pi/2\). Draw the branch cut of the logarithm. Does the branch cut intersect the region \( \Omega_2 \)? Is the logarithm complex differentiable along the branch cut? Why or why not. Was this a good choice for the branch of the logarithm to use?

d) Compute the real and imaginary parts of \( f(g(w)) \). State your results in terms of the polar coordinates \( \rho = |w|, \phi = \arg(w) \).
Figure 1:

e) Show that the real part of $f(g(w))$ solves the steady-state heat equation on $\Omega_2$, do so by showing that it satisfies the stated boundary conditions and is harmonic.
Part 2: Contour Integration and Cauchy Integral Formula.

a) Factor the function $h(z) = z^2 + 4$ into the form $(z - z_1)(z - z_2)$, where $z_1, z_2$ are its roots.

b) Compute the contour integral $\int_{C_1} \frac{1}{z-2i} dz$ using the parameterization $z(t) = 2i + e^{it}$ with $0 \leq t \leq 2\pi$, see Figure 2.

c) Compute the contour integral $\int_{C_2} \frac{1}{z+2i} dz$ using the parameterization $z(t) = -2i + e^{it}$ with $0 \leq t \leq 2\pi$, see Figure 2.

d) Compute the contour integral $\int_{C_3} \frac{1}{z^2+4} dz$ using the Cauchy Integral Formula. Do so by factoring the integrand into the form $\frac{1}{z^2+4} = \frac{A}{z-2i} + \frac{B}{z+2i}$, for some complex $A$ and $B$. Apply the formula separately to each of the resulting contour integrals. State whether or not the sum of the values obtained in parts (b), (c) give the same result.

a) Show that the contour integral over just the contour $C_R = \{ |z| = R, \text{Im}(z) > 0 \}$ in Figure 3 tends to zero as $R \to \infty$. Use the modulus bounds of a contour integral.

b) Show that $\lim_{R \to \infty} \int_{C} \frac{1}{z^2+4} \, dz = \int_{C'} \frac{1}{z^2+4} \, dz = \int_{-\infty}^{\infty} \frac{1}{x^2+4} \, dx$. Use part (a).

c) Give the value of the real-valued integral $\int_{-\infty}^{\infty} \frac{1}{x^2+4} \, dx$ using the results of part (b) and from part 2.

d) Give a general expression for the value of real-valued integrals of the form $\int_{-\infty}^{\infty} \frac{1}{x^2+m^2} \, dx$ $m \neq 0$ using your insights from parts (a-c).
Part 4: Potential Flows

The velocity field of a fluid represented in a 2D flow plane can be expressed in terms of a complex number \( V(x, y) = p(x, y) + iq(x, y) \). For irrotational inviscid fluid flows with translational planar symmetry in one direction, the flow is given by the gradient of a potential \( \phi(x, y) \) with velocity components \( V = \phi_x(x, y) + i\phi_y(x, y) \). When the fluid is incompressible the potential must be harmonic with \( \phi_{xx} + \phi_{yy} = 0 \). Let \( \psi(x, y) \) denote the harmonic conjugate of \( \phi(x, y) \). The fluid flow can then be represented in terms of a complex differentiable function \( F(z) = \phi(x, y) + i\psi(x, y) \) which is called the complex potential.

a) Show that the velocity of the fluid \( V = \phi_x(x, y) + i\phi_y(x, y) \) is obtained from the complex potential by \( V = \overline{F'(z)} \), where the bar denotes complex conjugation. Use that \( \phi \) and \( \psi \) are the real and imaginary parts of a complex differentiable function and must satisfy the Cauchy-Riemann equations.

b) Consider a uniform flow which has the constant velocity \( V(x, y) = a + ib \) for all points in the plane. Find the potential \( \phi \) for this flow. Find its harmonic conjugate. State the complex potential \( F \) for this flow.

c) Flows can be obtained for other geometries by using a composition of complex differentiable functions \( F(g(w)) \) which will then have real and imaginary parts which are also harmonic functions. Find a complex differentiable function \( g(w) \) which maps the wedge...
with angle $\pi/4$ to the upper half-plane, see Figure 4. Compute the potential $F(g(w))$ for the wedge, where $F$ is given in part (b) and take $b = 0$. Show that the velocity is tangent to the boundary of the wedge provided $b = 0$.

d) The function $\psi(x, y)$ is called the stream function of a flow. Its level curves $\psi(x, y) = c$ are called streamlines and show the path that tracers dropped into the fluid would follow when swept along by the flow. Sketch qualitatively a few of the streamlines for the flow in part (c) for the wedge being sure to orient the streamlines in the direction of the flow. Show that the boundaries of the wedge are streamlines.