1D Finite Element Method for Fourth Order Equations:

1) Consider the fourth order equation:
   
   \[ ru''''(x) - pu''(x) + qu(x) = f(x) \]
   
   \[ u(0) = g_1, u'(0) = g_2, u''(\pi) = 0, pu'(\pi) - ru'''(\pi) = 0. \]

   a) State the functional \( I(v) \) which gives a variational principle for the equation on the space \( \mathcal{H}_E^2 \). (Hint: Be sure to consider carefully the terms which arise in the integration by parts.)

   b) Take \( u(0) = g_1, u'(0) = g_2 \) to be the essential conditions. Show that the remaining boundary conditions of the equation correspond to natural conditions of the functional.

   c) For \( N \) equally spaced nodes on \([0, \pi]\) consider the Hermite Cubic Elements discussed in lecture and give explicit expressions for the piecewise cubic basis functions \( \Psi(x) \) and \( \omega(x) \).

   d) For the Hermite Cubic Elements write a subroutine to construct for any \( N \) the global bending matrix \( K_2 \), the global stiffness matrix \( K_1 \), and also the global mass matrix \( K_0 \) for the fourth order equation when \( p, q, r \) are all constant.

   e) Write a subroutine to construct the load vector \( \tilde{F} \) for \( f(x) \) approximated by its piecewise cubic interpolation \( f_I(x) \).

2) Consider the fourth order equation with \( r = p = 1, q = 0 \) with the load \( f(x) = 20 \cos(2x) - 4, u(0) = 1, u'(0) = -4\pi, u''(\pi) = 0, u'(\pi) - u'''(\pi) = 0. \)

   a) Write a code which computes the solution vector \( u^h \) of the finite element equation \( Ku^h = \tilde{F} \), where \( K = K_0 + K_1 + K_2 \).

   b) Plot the approximating solution \( u^h(x) \) by using the basis functions \( \Psi_j(x) \) and \( \omega_j(x) \) associated with each node \( x_j \). Give a plot of the solutions on the same graph for \( N = 4, N = 10, N = 100, N = 200 \). Plot the solutions at the points \( z_k \) with mesh spacing \( \Delta z = z_{k+1} - z_k = \pi/100000 \) on \([0, \pi]\). Also plot the analytic solution \( u(x) = \cos(2x) + 2x^2 - 4\pi x \).

   c) Estimate the order of accuracy of the method in each of the error norms \( \mathcal{H}^s \), \( s = 0, 1, 2 \) using the analytic solution. Estimate the error by computing a quadrature of the integrals appearing in each of the norms. This should be done using a method which incurs errors.
much smaller than that associated with the FEM errors being estimated. For this purpose composite Gaussian quadratures could be used, but if you are unfamiliar with these methods you may also use composite Simpson’s rule. If you use Simpson’s rule be sure that the quadrature nodes \( z_k \) have spacing \( \Delta z = z_{k+1} - z_k \ll h \), for our problem to be safe take at least \( \Delta z = \pi/100000 \). State the estimated order of accuracy \( h^\gamma \) of the FEM in terms of \( \gamma \). Discuss how the observed order of accuracy obtained for \( s = 0, 1, 2 \) for the Hermite Cubic Elements compares with what was discussed in the lecture.