Name: **Solution Key**

Directions: Please be sure to answer each question carefully in the space provided. Please present your work neatly so you can receive maximum credit. If you have any questions please feel free to ask.

Midterm Exam: November 8th, 2007
Professor: Paul J. Atzberger

Scoring:

Problem 1: ________________

Problem 2: ________________

Problem 3: ________________

Final Score: ________________
Problem 1:

a) Give a functional \( I(v) \) for the variational principle for the 1D steady-state heat equation:

\[
-u'' = f, \text{ for } 0 < x < \pi. \quad (1)
\]
\[
u(0) = 0, u'(\pi) = 0 \quad (2)
\]

\[
I(v) = \int_{0}^{\pi} (v')^2 \, dx - 2 \int_{0}^{\pi} f \, v \, dx.
\]

Verify: (variations of \( I \))

\[
I(v) \leq I(u + \epsilon v) \quad \text{for any } v \in \mathcal{V}, \quad v|_0 = 0,
\]

\[
= I(u) + \epsilon \int_{0}^{\pi} \left[ \int_{0}^{\pi} u' v' \, dx - \int_{0}^{\pi} f v \, dx \right] \, dx + \epsilon^2 \left[ \int_{0}^{\pi} (v')^2 \, dx \right]
\]

Since \( \epsilon \) is arbitrary

\[
\int_{0}^{\pi} u' v' \, dx - \int_{0}^{\pi} f v \, dx = 0, \quad \forall v \in \mathcal{V}, \quad v|_0 = 0.
\]

Integration by parts gives

\[
\int_{0}^{\pi} (-u'' - f) \, v \, dx + u'(\pi) v(\pi) = 0, \quad \forall v \in \mathcal{V}, \quad v|_0 = 0.
\]

By choosing \( v \) to approximate the Dirac \( \delta \)-function at \( x_0 \in (0, \pi) \), it can be shown

\[
- u''(x_0) - f(\pi) = 0 \quad \text{holds.}
\]

By choosing \( v \) to be one at \( \pi \), \( v|_\pi = 1 \)

and vanish outside a small layer of the boundary at \( \pi \), it can be shown

\[
u'(\pi) = 0.
\]

Therefore, if the minimizer was in \( \mathcal{V}^1 \) it must satisfy the PDE and bnd. cond. If it is only in \( \mathcal{V}^1 \), then above shows \( u \) satisfies weak form PDE.
b) Over what space do the derivatives appearing in the functional suggest need to be used for minimization? Give your answer in terms of $H^2$.

The functional will have finite values if $v \in H^1$.

c) For this PDE and functional $I(v)$ state what are the natural and essential boundary conditions.

$\nabla (0) = 0$ are essential,

$\nabla (\ell) = 0$ are natural.

d) How are essential boundary conditions imposed? Why are they necessary?

Essential bnd. cond. imposed by constraining the admissible space of functions over which minimization is taken.

Natural bnd. cond. are imposed by the form of the functional.

e) What if instead of a homogenous Neumann boundary condition at $\pi$ we instead want $u'(\pi) = b$? Give a modified functional which has minimizers satisfying this condition.

$$\tilde{I}(v) = I(v) + (-2v'(\pi)b).$$

$$\frac{\delta I}{\delta v} = \frac{\delta \tilde{I}}{\delta v} + -2b v'(\pi)$$

$$= 2 \int_0^\ell (-u'' - f) v \, dx + 2 \left( v'(\pi)b - b v(\ell) \right) = 0$$

$\forall v \in H^1 \text{ s.t. } v(0) = 0$.

Arguing as before this requires a minimizer to satisfy $u'(\ell) - b = 0$. 
Problem 2:
The Ritz Finite Element Methods are obtained by performing a minimization of the functional appearing in the variational principle for the PDE over some finite dimensional subspace. In this problem you will derive a FEM for the PDE in equation 1. For ease of the calculations suppose throughout we use the space $S^h$ with piecewise linear continuous functions on only three nodes $x_0 = 0, x_1 = \pi/3, x_2 = 2\pi/3, x_3 = \pi$.

a) Calculate the element stiffness matrix for $\text{interval } [x_1, x_2]$.

\[
\int_{x_1}^{x_2} (q_1' n')^2 \, dx = q_1^T K_1 q_1
\]

\[
\int_{x_1}^{x_2} (q_2' n')^2 \, dx = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}^T \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
h = x_2 - x_1 = \frac{\pi}{3},
\]

\[
\psi_j(x) = \sum_{j=1}^{3} q_j \psi_j(x), \quad \psi_j(x) = \begin{cases} \frac{x}{h} - \frac{j-1}{h}, & jh \leq x < (j+1)h \\ 0, & \text{otherwise} \end{cases}
\]

\[
\psi_j'(x) = \begin{cases} \frac{1}{h}, & jh \leq x < (j+1)h \\ -\frac{1}{h}, & (j-1)h \leq x < jh \end{cases}
\]

\[
\int_{x_1}^{x_2} \left( q_2 \left( \frac{1}{h} \right)^2 + q_1 \left( -\frac{1}{h} \right)^2 \right) \, dx
\]

\[
= \int_{x_1}^{x_2} q_2 \left( \frac{1}{h} \right)^2 - 2q_1 q_2 \left( \frac{1}{h} \right)^2 + q_1 \left( \frac{1}{h} \right)^2 \, dx
\]

\[
= q_2 \left( \frac{1}{h} \right)^2 \cdot h - 2q_1 q_2 \left( \frac{1}{h} \right)^2 \cdot h + q_1 \left( \frac{1}{h} \right)^2 \cdot h
\]

\[
= q_2 \left( \frac{1}{h} \right)^2 + 2q_1 q_2 \left( \frac{1}{h} \right)^2 + q_1 \left( \frac{1}{h} \right)^2
\]

\[
= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{h} \\ \frac{1}{h} \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
K_1 \text{ element stiffness matrix for EXW.}
\]
b) Calculate the global stiffness matrix

\[
S_{x_0} (\cdot v h')^+ dx = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix},
\]

\[
S_{x_1} (\cdot v h')^+ dx = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},
\]

\[
S_{x_3} (\cdot v h')^+ dx = \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}.
\]

This shows that

\[
S_{x_0} (\cdot v h') dx = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.
\]

The boundary condition \( u(0) = 0 \), requires \( q_0 = 0 \). This gives the global stiffness matrix for our problem

\[
K_1 = h \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.
\]
c) State the Finite Element equations for equation 1.

\[ k_1 q = F \]

\[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix}
= 
\begin{pmatrix}
S_{x_0}^{x_1} \frac{1}{n} f(x) dx - S_{x_1}^{x_2} \frac{1}{n} f(x) dx \\
S_{x_1}^{x_2} \frac{1}{n} f(x) dx - S_{x_2}^{x_3} \frac{1}{n} f(x) dx \\
S_{x_2}^{x_3} \frac{1}{n} f(x) dx
\end{pmatrix}
\]

This can be simplified further.

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix}
= 
\begin{pmatrix}
S_{x_0}^{x_1} f(x) dx - S_{x_1}^{x_2} f(x) dx \\
S_{x_1}^{x_2} f(x) dx - S_{x_2}^{x_3} f(x) dx \\
S_{x_2}^{x_3} f(x) dx
\end{pmatrix}
\]

In practice, the (load vector) \( F \) must be approximated to sufficient accuracy, in this case at least order \( n^0 \).
**Problem 3:** For the Finite Element Method with linear elements we derived rigorous error estimates.

a) State the error estimates obtained for \( e^h = u^h - u \) in the norms, \( \| e^h \|_0, \| e^h \|_1, \) and energy norm \( \sqrt{a(e^h, e^h)} \).

\[
\begin{align*}
\| e^h \|_0 & \leq C h^2 \\
\| e^h \|_1 & \leq C h \\
\sqrt{a(e^h, e^h)} & \leq C h
\end{align*}
\]

b) If it were only known that the FEM converges in energy norm would this be sufficient for it to converge in the \( H^1 \)-norm? In other words, if the energy norm of the error goes to zero is this sufficient to ensure that the strains of the FEM solution converge to the strains of the exact solution? Why or why not?

Yes, since the energy norm and \( H^1 \) norm are equivalent for PDE (1). Equivalence of norms means \( \sigma \| e^h \|_1 \leq \sqrt{a(e^h, e^h)} \leq K \| e^h \|_1 \), for some constants \( \sigma, K \). Therefore, convergence in one norm implies convergence in the other.

c) For linear elements to what order of accuracy must we perform the quadrature when computing the approximate load vector to ensure that the FEM retains its theoretical order of accuracy?

We must use a quadrature of at least order \( h^2 \) not to dominate the FEM errors expected from the theoretical analysis.