Contingent Claims and the Arbitrage Theorem

Paul J. Atzberger

Please send any comments to:
Paul J. Atzberger
atzberg@math.ucsb.edu

Introduction

No arbitrage principles play a central role in models of finance and economics. The assumption of no arbitrage essentially states that there is "no free lunch" in a market. In other words, there are no zero cost investment strategies that allow for a market participant to make a profit without taking some risk of a loss. We shall mainly be interested in pricing derivative contracts, which derive their value from an underlying asset or index such as a stock or interest rate. In this context no arbitrage can be used to determine a price for a contract by constructing portfolios which replicate the payoff in complete markets or give a bound on permissible prices for a contract by constructing portfolios which bound the payoff in incomplete markets.

Using results from linear programming, the pricing theory obtained by constructing such replicating portfolios can be shown to be have as its dual problem the construction of a risk-neutral probability for the market. The "risk-neutral" terminology can be motivated by considering an individual who values a contract by only considering its expected payoff and is uninfluenced by the riskiness of obtaining an uncertain payoff. Such an individual we might call a "risk-neutral" investor. The risk-neutral probability of a market is defined as a probability measure which when taking the expected payoff of any derivative contract and discounting into todays dollars gives the price of the contingent claim consistent with no arbitrage.

The connection between the existence of risk-neutral probability measures and no arbitrage in a market is established in the The Arbitrage Theorem. For complete markets the Arbitrage Theorem states that there is no arbitrage if and only if there exists a risk-neutral probability measure. In other words there is no arbitrage in a market only if the derivatives have a price which is given by the expectation of its payoff, under the risk-neutral probability measure, discounted to express the value in today's dollars. This theory plays a central role in the pricing of options in mathematical finance by connecting arbitrage considerations with probability theory. This theory will be the main subject of these notes.

The materials presented here are taken from the following sources: *Derivatives Securities: Lecture Notes* by R. Kohn, *Options, Futures, and Other Derivatives* by Hull, and *Derivatives Securities* by Jarrow and Turnbill.

One Period Markets

We shall consider price movements over a single time period $[0, T]$ for a market consisting of two assets, a stock and a bond. Many models could in principle be used to model the price movements, we shall restrict ourselves in these notes to two very simple models, in particular, a binomial and trinomial tree to illustrate the basic ideas. Throughout the discussion we shall assume that the interest rate for the bond is fixed with rate r.

Binomial Market Model

Let the price of the stock at time 0 be denoted by s_0. In the binomial market model we shall allow for only two possibilities for the movement of the price over the period $[0, T]$. Let these be denoted by s_1, s_2 with $s_1 < s_2$, see figure 1.

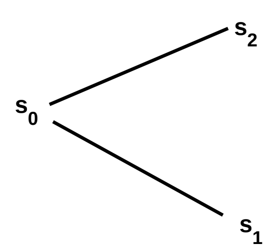

Figure 1: Binomial Model

In our model no arbitrage requires that there be no zero cost investment strategies that would allow for an investor to make a profit without some risk of a loss. This places constraints on the possible values of s_1, s_2 in our model. In particular, if we had $s_1 > e^{rT}s_0$ and $s_2 > e^{rT}s_0$ then there would be an arbitrage opportunity in the market. We could ensure ourselves a profit by taking out a bank loan for the amount s_0 at time 0 and buying the stock. At time T we would then owe the bank $s_0 e^{rT}$, which is less than we would make from selling the stock in either outcome of the market, s_1 or s_2. A similar type of investment strategy can be constructed if $s_1 < e^{rT}s_0$ and $s_2 < e^{rT}s_0$, except in this case we sell the stock and buy a bond. From the assumption of no arbitrage the prices in our model must satisfy:

$$s_1 < s_0 e^{rT} < s_2.$$

We shall now consider the pricing of a derivative security which has a payoff which depends only on the price of the stock at time T. We shall denote the prices of the contract by f_0, f_1, f_2 with the same indexing convention we used for the stock. For example, a call option with strike price K would have the value at time T given by $f_1 = (s_1 - K)_+$ and $f_2 = (s_2 - K)_+$. A put option would have the value $f_1 = (K - s_1)_+$ and $f_2 = (K - s_2)_+$. Determining the value of the contract at time 0 requires a bit more work, which we shall now discuss.

The basic strategy to determine the value of the contingent claim at time 0 is to build a portfolio of the bond and stock which gives the same payoff as the contract. No arbitrage then requires that the contract be worth at time 0 the same as the replicating portfolio.

We shall denote the weights for the assets in our portfolio by w_1 for the stock and w_2 for the bond, so that the portfolio at time zero has the value:

$$w_1 S_0 + w_2 e^{-rT}.$$

Recall a bond which pays \$1 at time T is worth e^{-rT} today.

Replicating the payoff of the contingent claim at time T requires that we solve the following linear equations for w_1, w_2:

$$\begin{aligned} w_1 s_1 + w_2 &= f_1 \\ w_1 s_2 + w_2 &= f_2. \end{aligned}$$

This has solution:

$$\begin{aligned} w_1^* &= \frac{f_2 - f_1}{s_2 - s_1} \\ w_2^* &= \frac{s_2 f_1 - s_1 f_2}{s_2 - s_1}. \end{aligned}$$

Since the portfolio with the weights w_1^* and w_2^* has the same payoff as the contingent claim in each outcome of the stock market, by no arbitrage the value of the contingent claim at time 0 must be:

$$V(f) = w_1^* s_0 + w_2^* e^{-rT}.$$

Markets in which for any contingent claim a replicating portfolio can be constructed are referred to as *complete markets*.

By using the definition of w_1^* and w_2^* and factoring the coefficients into common terms of f_1 and f_2 we have:

$$V(f) = e^{-rT}\left[(1-q)f_1 + q f_2\right]$$

where $q = \frac{s_0 e^{rT} - s_1}{s2 - s1}$. It can be readily checked that no arbitrage holds for the market, $s_1 < s_0 e^{rT} < s_2$, if and only if $0 < q < 1$. Thus we see that the Arbitrage Theorem holds for the binomial market model.

Trinomial Market Model

Let us now consider a market which at time T has three different outcomes for the stock prices $s_1 < s_2 < s_3$, see figure 2. The assumption of no arbitrage in this market requires that at least one outcome does better than the risk-free return and that at least one outcome does worse. This requires:

$$s_1 < s_0 e^{rT}, \text{ and, } s_0 e^{rT} < s_3.$$

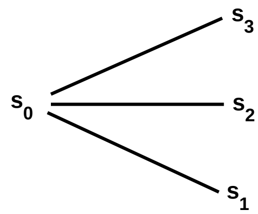

Figure 2: Trinomial Model

Consider again a contingent claim with value f_0 at time 0 and value f_1, f_2, f_3 at time T. For a given outcome of the stock market, we again know the values f_1, f_2, f_3 of the contract, but do not know the value of f_0 at time 0. We again seek to construct a "replicating" portfolio. In this case we must find the weights w_1, w_2 by solving the following linear system for the payoff at time T:

$$\begin{aligned} w_1 s_1 + w_2 &= f_1 \\ w_1 s_2 + w_2 &= f_2 \\ w_1 s_3 + w_2 &= f_3. \end{aligned}$$

We see that this linear system only permits a solution for a select subset of contingent claims with payoffs $[f_1, f_2, f_3]$ within a 2-dimensional subspace. Thus the payoff of many of the contingent claims can not be replicated by formulating a portfolio using only the stock and bond. Such markets are referred to as *incomplete markets*.

While we can not determine an exact price by this method, useful information can still be obtained. Any portfolio having a larger value at time T than the contingent claim's payoff, in each outcome of the stock market, must have a greater value at time 0 by no arbitrage arguments. A similar statement follows for portfolios having lesser value at time T than the contingent claim's payoff. This gives the bounds:

$$\begin{aligned} V(f) &\leq w_1^+ + w_2^+ e^{-rT} \\ V(f) &\geq w_1^- + w_2^- e^{-rT} \end{aligned}$$

where the superscript $+$ indicates that the portfolio dominates the payoff of the contingent claim, while $-$ indicates the portfolio has a smaller value than the payoff of the contingent claim in each outcome of the stock market.

More precisely, we can express this as:

$$\min_{\substack{w_1 s_i + w_2 \leq f_i \\ i=1,2,3}} w_1 s_0 + w_2 e^{-rT} \leq V(f) \leq \min_{\substack{w_1 s_i + w_2 \leq f_i \\ i=1,2,3}} w_1 s_0 + w_2 e^{-rT}.$$

General One Period Market Models

We shall use the following notation to express the structure of a market in general. Let the payoff of asset i for outcome α of the market be denoted by payoff matrix $D_{i,\alpha}$. Let the price of the i^{th} asset at time 0 be denoted by p_i.

For example, the payoff matrix for the assets in the *binomial market* is:

$$D = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}$$

with asset prices at time 0:

$$p = \begin{bmatrix} s_0 & e^{-rT} \end{bmatrix}^T.$$

For the *trinomial market* we have:

$$D = \begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \end{bmatrix}$$

and asset prices:

$$p = \begin{bmatrix} s_0 & e^{-rT} \end{bmatrix}^T.$$

In a general market with N assets with M possible outcomes we have:

$$D = \begin{bmatrix} 1 & \cdots & 1 \\ D_{2,1} & \cdots & D_{2,1} \\ \vdots & & \vdots \\ D_{N,1} & \cdots & D_{N,M} \end{bmatrix}$$

$$p \in \mathbb{R}^N.$$

The Duality of Replicating Portfolios and Risk-Neutral Probabilities

The problem of constructing the best replicating portfolio to determine the price of a contingent claim turns out to be dual to the problem of finding the best valuation of the contingent claim under all risk-neutral probabilities permitted by the market. In particular, if we consider the upper bound on the price of the contingent claim in a market with N general assets we can express the problem of finding the weights $\{w_i\}$ of the best replicating portfolio as:

$$\begin{aligned} V(f) &\leq \min_{\sum_i w_i D_{i,\alpha} \geq f_\alpha} \sum_i w_i p_i \\ &= \min_{w_i} \max_{\pi_\alpha \geq 0} \sum_i w_i p_i + \sum_\alpha \pi_\alpha (f_\alpha - \sum_i w_i D_{i,\alpha}) \\ &= \max_{\pi_i \geq 0} \min_{w_i} \sum_i w_i p_i + \sum_\alpha \pi_\alpha (f_\alpha - \sum_i w_i D_{i,\alpha}) \\ &= \max_{\pi_\alpha \geq 0} \min_{w_i} \sum_i w_i (p_i - \sum_\alpha \pi_\alpha D_{i,\alpha}) + \sum_\alpha \pi_\alpha f_\alpha \\ &= \max_{\substack{\sum_\alpha \pi_\alpha D_{i,\alpha} = p_i \\ \pi_\alpha \geq 0}} \sum_\alpha \pi_\alpha f_\alpha. \end{aligned}$$

The first equality follows by noting that:

$$\max_{\pi_\alpha \geq 0} \sum_\alpha \pi_\alpha (f_\alpha - \sum_i w_i D_{i,\alpha}) = \begin{cases} 0, \text{ if } \sum_i D_{i,\alpha} \geq f_\alpha \text{ for all } \alpha \\ \infty, \text{ +otherwise} \end{cases}$$

The second inequality in which we used that $\min\max = \max\min$ follows from the duality theorem.

For the lower bound a similar argument can be made to obtain:

$$\begin{aligned} V(f) &\geq \max_{\sum_i w_i D_{i,\alpha} \geq f_\alpha} \sum_i w_i p_i \\ &= \min_{\substack{\sum_\alpha \pi_\alpha D_{i,\alpha} = p_i \\ \pi_\alpha \geq 0}} \sum_\alpha \pi_\alpha f_\alpha. \end{aligned}$$

This gives the following bound on the price of a contingent claim:

$$\min_{\substack{\sum_\alpha \pi_\alpha D_{i,\alpha} = p_i \\ \pi_\alpha \geq 0}} \sum_\alpha \pi_\alpha f_\alpha \leq V(f) \leq \max_{\substack{\sum_\alpha \pi_\alpha D_{i,\alpha} = p_i \\ \pi_\alpha \geq 0}} \sum_\alpha \pi_\alpha f_\alpha.$$

Now if we assume that one of the assets in the market is a risk-free bond with interest rate r, say the asset with index $i = 1$, then we have $e^{-rT} = \sum_\alpha D_{1,\alpha} \pi_\alpha$. Since the first row of the payoff matrix is of the form $D_{1,\alpha} = 1$ and we have $e^{-rT} = \sum_\alpha \pi_\alpha$, if we define $\hat{\pi}_\alpha = e^{rT} \pi_\alpha$ then the weights $\{\hat{\pi}_\alpha\}$ are a risk-neutral probability measure.

This allows for us to express the inequalities as:

$$\min_{\substack{\text{risk-neutral} \\ \text{probability } \hat{\pi}}} e^{-rT} \sum_\alpha \hat{\pi}_\alpha f_\alpha \leq V(f) \leq \max_{\substack{\text{risk-neutral} \\ \text{probability } \hat{\pi}}} e^{-rT} \sum_\alpha \hat{\pi}_\alpha f_\alpha.$$

For a complete market this becomes an equality and we see that the risk-neutral probability terms are precisely the dual values of the weights of the replicating portfolio.

The General Principle of No Arbitrage

For the general class of markets considered here, no arbitrage corresponds to the following:
Principle of No Arbitrage:

(a) $\sum_{i=1}^N w_i D_{i,\alpha} \geq 0$ for all α, implies $\sum_{i=1}^N w_i p_i \geq 0$.

(b) if we have $\sum_{i=1}^N w_i D_{i,\alpha} \geq 0$ for all α and $\sum_{i=1}^N w_i p_i = 0$ then we must have $\sum_{i=1}^N w_i D_{i,\alpha} = 0$.

The condition (a) states that if the payoff is non-negative then the value of the portfolio must be non-negative. The statement (b) says that if the payoff of a portfolio is always non-negative but costs nothing then the non-negative payoff must be the trivial payoff which is zero. Part (a) is sometimes referred to as the "weak no arbitrage principle" whereas (a) and (b) together is referred to as the "strong no arbitrage principle".

The Arbitrage Theorem

Theorem 1 *(Arbitrage Theorem): The market satisfies (a) if and only if there exists $\pi_\alpha \geq 0$ such that*

$$\sum_\alpha D_{i,\alpha} \pi_\alpha = p_i, \;\; i = 1, \ldots, N. \tag{1}$$

The market satisfies both (a) and (b) if in addition the π_α can be chosen to be all strictly positive.

Before proving the theorem we remark that since the weights π_α are non-negative they can be renormalized to form probability weights (provided they are not all zero). Thus in essence, the theorem states that no arbitrage in a market is equivalent to the existence of risk-neutral probability weights for the assets.

sketch of the proof:

One direction of the theorem follows rather easily. If there are non-negative weights $\pi_\alpha \geq 0$ so that equation 1 holds then if $\sum_i w_i D_{i,\alpha} \geq 0$ for all α it follows immediately that $\sum_\alpha \sum_i w_i D_{i,\alpha} \pi_\alpha = \sum_i w_i p_i \geq 0$, which shows (a). If the weights are positive $\pi_\alpha > 0$ then (b) follows by a similar argument.

To prove the other direction of the theorem we shall make use of Farkas's Lemma which states that: $A^T y \geq 0 \Rightarrow b^T y \geq 0$, $\forall y$ holds if and only if there is a solution $x \geq 0$ satisfying $Ax = b$. In other words, this lemma states that if a collection of inequalities implies another inequality, this occurs in a rather trivial fashion. In particular, the new inequality is a positive linear combination of the inequalities in the collection.

We shall now show that the no arbitrage condition (a) implies the existence of the non-negative weights. Condition (a) can be expressed in vector notation as stating: $D^T w \geq 0$ for all w implies $w^T p \geq 0$. By Farkas's Lemma we have that there is a solution $\pi \geq 0$ satisfying $D\pi = p$. This shows that condition (a) implies the existence of the non-negative weights satisfying equation 1.

We shall now show that if both condition (a) and (b) hold then weights π_α can be found which are strictly positive. To do this we shall label the weights so that $\pi_1, \ldots, \pi_{M'} > 0$ and $\pi_{M'+1}, \ldots, \pi_M = 0$. If all the weights are already positive we are finished, so we shall only consider the case when $M' < M$.

In the case that there are positive weights $a_\alpha > 0$ for some set of coefficients b_α such that:

$$\sum_{\alpha=M'+1}^{M} a_\alpha D_{\cdot,\alpha} = \sum_{\alpha=1}^{M'} b_\alpha D_{\cdot,\alpha} \tag{2}$$

we have:

$$\begin{aligned} p_i &= \sum_{\alpha=1}^{M'} D_{i,\alpha} \pi_\alpha \\ &= \epsilon \sum_{\alpha=M'+1}^{M} a_\alpha D_{i,\alpha} + \sum_{\alpha=1}^{M'} D_{i,\alpha}(\pi_\alpha - \epsilon b_\alpha). \end{aligned}$$

The equality on the second line follows by adding and subtracting the term on the left hand side of equation 2 and then substituting the right hand side in the second summand.

We can obtain positive coefficients $\tilde{\pi}_\alpha$ in this case by directly using that $a_\alpha > 0$. In particular, we have that $\tilde{\pi}_\alpha = \epsilon a_\alpha > 0$, for $\alpha = M' + 1, \ldots, M$. To obtain positive coefficients for the remaining indices we use that $\pi_\alpha > 0$ and make the factor $\epsilon > 0$ sufficiently small so that $\tilde{\pi}_\alpha = \pi_\alpha - \epsilon b_\alpha > 0$ for $\alpha = 1, \ldots, M'$.

Now if there are not positive weights a_α for any coefficients b_α then the following statement holds:

$$\sum_{\alpha=M'+1}^{M} a_\alpha D_{\cdot,\alpha} = \sum_{\alpha=1}^{M'} b_\alpha D_{\cdot,\alpha}, \; a_\alpha \geq 0 \Rightarrow a_\alpha = 0, \alpha = M' + 1, \ldots, M.$$

This states that the space spanned by the vectors $\{D_{\cdot,1}, \ldots, D_{\cdot,M'}\}$ does not include any non-trivial non-negative linear combination of the vectors $\{D_{\cdot,M'+1}, \ldots, D_{\cdot,M}\}$.

Geometrically, this corresponds to the set $\{D_{\cdot,M'+1}, \ldots, D_{\cdot,M}\}$ forming a cone which intersects the linear space spanned by $\{D_{\cdot,1}, \ldots, D_{\cdot,M'}\}$ only at the vertex point $\mathbf{0}$. From this it can be shown that there is a vector w which is orthogonal to the linear space $\{D_{\cdot,1}, \ldots, D_{\cdot,M'}\}$ and which for the half space $\{x \in \mathbb{R}^N | \langle x, w \rangle > 0\}$ contains the cone. In other words, there is a w such that:

$$\begin{aligned} \langle D_{\cdot,\alpha}, w \rangle &= 0, \; \alpha = 1, \ldots, M' \\ \langle D_{\cdot,\alpha}, w \rangle &> 0, \; \alpha = M' + 1, \ldots, M. \end{aligned}$$

Writing this out in terms of summands we have:

$$\begin{aligned}\sum_i w_i D_{i,\alpha} &= 0,\ \alpha = 1, \ldots, M' \\ \sum_i w_i D_{i,\alpha} &> 0,\ \alpha = M'+1, \ldots, M.\end{aligned}$$

But then w represents a portfolio that has some positive payoffs without any risk of a loss, and requires zero cost of investment since $\sum_i w_i p_i = \sum_i \sum_{\alpha=1}^{M'} w_i D_{i,\alpha} \pi_\alpha = 0$. This contradicts the assumption of no arbitrage. Therefore this second case, in which there are no positive weights a_α, is excluded by the no arbitrage conditions (a) and (b).

■

References

[1] Convex Analysis and Nonlinear Optimization by Borwein and Lewis, Canadian Mathematical Society
[2] Numerical Optimization by Nocedal and Wright
[3] Derivatives Securities: Lecture Notes by R. Kohn.
[4] Options, Futures, and Other Derivatives by Hull.
[5] Derivatives Securities by Jarrow and Turnbill.

Please send any comments to:
Paul J. Atzberger
atzberg@math.ucsb.edu