Elasticity Theory

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206D: Finite Element Methods
University of California Santa Barbara

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The **symmetric gradient approximation** for strain is denoted by

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The symbol $\wedge$ denotes the vector cross-product in $\mathbb{R}^3$.

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Typically we will have $F = \nabla\phi(\mathbf{x})$.

Elastic Materials:

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where for every deformed state

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$$\tilde{\Sigma}(C) = \tilde{\Sigma}(I + 2E) = -\pi I + \lambda\,\mathrm{trace}(E) I + 2\mu E + o(E), \quad \text{as } E \to 0.$$

Proof (sketch):
From definition of $\iota_j(A)$ we have $\iota_j(E) = O(E^j)$. From smoothness of γ_i, we have only $\iota_1(E)$ contributes in the expressions. From $C = I + 2E$, $C^2 = I + 4E + o(E)$, to obtain leading order we expand as $\gamma_0(E) = a_0 + b_1\iota_1(E) + o(E)$, $\gamma_1(E) = a_1 + O(E)$, $\gamma_2(E) = a_2 + O(E)$. This yields

$$\tilde{\Sigma}(C) = (a_0 + a_1 + a_2)I + b_1\iota_1(E)I + (2a_1 + 4a_2)E + o(E).$$

Using $\iota_1(E) = \mathrm{trace}(E)$ the result follows. ■
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Remark: For small deformations, if we replace linearization in E with linearization in ϵ approach is called **geometrically linear theory**.

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Remark: Results in saddle-point problems.

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Displacement Formulation

Variational Principle for Displacement Formulation

$$\Pi[v] = \int_\Omega \left(\mu\, \epsilon[v] : \epsilon[v] + \frac{\lambda}{2} \left(\mathrm{div}(v)\right)^2 - f \cdot v \right) dV_{\mathbf{x}} + \int_{\Gamma_1} g \cdot v \, dA_{\mathbf{x}} \to \min$$

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Allow typically more accurate calculation of stresses since represented directly as degrees of freedom.

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Useful in establishing that variational problems involving strain are elliptic.

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There are results establishing conditions for well-posedness for the other formulations. These typically involve analysis establishing the Babuska-Brezzi inf-sup conditions hold (discussed with mixed method theory).

Locking Phenomena

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In the nearly incompressible regime, referred to as **volume locking** or **Poisson locking**.

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Discretization: Choose appropriate finite element spaces for the mixed method (discussed in other lecture).