Elasticity Theory

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206D: Finite Element Methods
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A material is modeled by a **reference configuration** $\tilde{\Omega}$ which is a closed bounded set in $\mathbb{R}^3$. The current configuration of the material body is described by the deformation mapping $\phi: \tilde{\Omega} \rightarrow \mathbb{R}^3$, assumed $\det \nabla \phi > 0$. The $\phi(x)$ represents the current position of the material point $x$ from the reference configuration. The displacement $u$ of the material is $u(x) = \phi(x) - x$. Very useful when modeling small deformations allowing for expansions neglecting higher orders. The deformation gradient is given by

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\nabla \phi = \begin{bmatrix}
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\end{bmatrix}.
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This allows us to express variations in the deformation with position as $\phi(x + z) - \phi(x) = \nabla \phi(x) \cdot z + o(z)$. The Euclidean distance between deformations to leading order is

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\| \phi(x + z) - \phi(x) \|_2 = \| \nabla \phi \cdot z + o(z) \|_2 \Rightarrow C := \nabla \phi^T \nabla \phi.
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\[ E := \frac{1}{2} (C - I). \]
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\]
In practice, the second quadratic term is often neglected to obtain an approximation.
The symmetric gradient approximation for strain is denoted by
\[
\epsilon_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\]
Assumptions for Equilibrium

Interactions of the body with the outside world is assumed to occur only through two types of applied forces:

(a) surface forces $t(x, n) \, dA$.

(b) body forces $f(x) \, dV$.

The $t(x, n)$ is called the Cauchy stress vector. The $f(x)$ is called the body force.

Axiom of Static Equilibrium

For a body $B$ in a deformed configuration at mechanical equilibrium, it is assumed that there exists a stress vector field $t$ so that for every smooth volume $V$ of $B$ we have

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Notational Conventions

- $M^3$: the set of $3 \times 3$ matrices.
- $M^3_+$: the set of $M^3$ with positive determinant.
- $O^3$: the set of orthogonal $3 \times 3$ matrices.
- $O^3_+$: the set $O^3 \cap M^3_+$.
- $S^3$: the set of symmetric $3 \times 3$ matrices.
- $S^3_+$: the set of positive definite matrices of $S^3$. 
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**Notational Conventions**

- $\mathbb{M}^3$ : the set of $3 \times 3$ matrices.
- $\mathbb{M}_+^3$ : the set of $\mathbb{M}^3$ with positive determinant.
- $\mathbb{O}^3$ : the set of orthogonal $3 \times 3$ matrices.
- $\mathbb{O}_+^3$ : the set $\mathbb{O}^3 \cap \mathbb{M}_+^3$. 
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<table>
<thead>
<tr>
<th>Symbol</th>
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## Notational Conventions

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\begin{align*}
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Consider \( t(\cdot, \mathbf{n}) \in C^1(B, \mathbb{R}^3) \), \( t(x, \cdot) \in C^0(S^2, \mathbb{R}^3) \), and \( f(x) \in C^0(B, \mathbb{R}^3) \).

There exists a symmetric tensor field \( T \in C^1(B, S^2) \) satisfying

\[
(i) \quad t(x, \mathbf{n}) = T(x) \mathbf{n}, \quad x \in B, \ \mathbf{n} \in S^2,
\]

\[
(ii) \quad T(x) = T^T(x), \quad x \in B,
\]

\[
(iii) \quad \text{div} \ T(x) + f(x) = 0, \quad x \in B.
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This \( T \) is called the Cauchy stress tensor.

This follows readily by using the Axiom of Static Equilibrium and the Gauss Divergence Theorem:

\[
\int_V f(x) \, dV + \int_{\partial V} T(x) \mathbf{n} \, dA = \int_V (f(x) + \text{div} \ T(x)) \, dV = 0.
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Can express mechanics either in deformed material body coordinates \( x \in \mathbb{R}^3 \) or in reference body frame \( x_R \in \bar{\Omega} \).

Transformations to reference configuration \( \bar{\Omega} \):

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Elasticity Theory

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Consider $\mathbf{t}(\cdot, \mathbf{n}) \in C^1(\mathcal{B}, \mathbb{R}^3)$, $\mathbf{t}(\mathbf{x}, \cdot) \in C^0(S^2, \mathbb{R}^3)$, and $\mathbf{f}(\mathbf{x}) \in C^0(\mathcal{B}, \mathbb{R}^3)$. There exists a symmetric tensor field $\mathbf{T} \in C^1(\mathcal{B}, S^2)$ satisfying

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\int_{\mathcal{V}} \mathbf{f}(\mathbf{x}) dV + \int_{\partial \mathcal{V}} \mathbf{T}(\mathbf{x})\mathbf{n} dA = \int_{\mathcal{V}} (\mathbf{f}(\mathbf{x}) + \text{div} \mathbf{T}(\mathbf{x})) dV = 0.
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Elasticity Theory

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Piola Transform:

- First Piola-Kirchhoff Stress Tensor $T_R$:
  $T_R := \det(\nabla \phi)T(\nabla \phi)$.

- Second Piola-Kirchhoff Stress Tensor $\Sigma_R$:
  $\Sigma_R := (\nabla \phi)^{-1}T_R = (\nabla \phi)^{-1}\det(\nabla \phi)T(\nabla \phi) - T_R$.

The Second Piola-Kirchhoff Stress Tensor $\Sigma_R$ is motivated by making a tensor that is symmetric. For small deformations, the three tensors $T$, $T_R$, $\Sigma_R$ become the same to leading order.
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Stress balance in reference configuration is
Elasticity Theory

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We define two associated stress tensors:
Elasticity Theory

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Elastic Materials:

A material is called elastic if there exists a mapping for the stress of the form 

\[ \hat{T} : M_3^+ \rightarrow S_3^+ \]

where for every deformed state 

\[ T(x) = \hat{T}(\nabla \phi(x_R)) \]

\( \hat{T} \) is the response function for the Cauchy stress for the material.

The \( T = \hat{T}(\nabla \phi(x_R)) \) is the constitutive equation for the material.

Transforming the tensors, we have the Piola-Kirchhoff stress 

\[ \hat{\Sigma}(F) := \det(F) F^{-1} \hat{T}(F) F^{-T} \]

Typically we will have 

\( F = \nabla \phi(x_R) \).

A material is called homogeneous if \( \hat{T} \) does not depend on \( x \).
Elastic Materials:

A material is called **elastic** if there exists a mapping for the stress of the form

\[
\hat{T} : M^{3} + \rightarrow S^{3},
\]

where for every deformed state \( T(x) = \hat{T}(\nabla \phi(x)) \). The \( \hat{T} \) is the response function for the Cauchy stress for the material. The \( T = \hat{T}(\nabla \phi(x)) \) is the constitutive equation for the material. Transforming the tensors, we have the Piola-Kirchhoff stress \( \hat{\Sigma}(F) := \text{det}(F) F^{-1} \hat{T}(F) F^{-T} \). Typically we will have \( F = \nabla \phi(x) \). A material is called **homogeneous** if \( \hat{T} \) does not depend on \( x \).
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\[ \hat{T} : \mathbb{M}_+ \rightarrow \mathbb{S}_+ , \]

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Typically we will have $F = \nabla \phi(x)$.

A material is called **homogeneous** is $\hat{T}$ does not depend on $x$. 
Given physical invariances, we make the assumption that the Cauchy stress vector \( \sigma(x, n) = T(x) n \) is independent of the choice of coordinates in the sense

\[
Q \sigma(x, n) = \sigma(Qx, Qn), \quad \forall Q \in O^3.
\]

A material that is frame-indifferent is referred to as an objective material.

Theorem
When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation \( Q \in O^3 \) that

\[
\hat{T}(QF) = Q \hat{T}(F) Q^T.
\]

We also have there exists a mapping \( \hat{\Sigma} : S^3 \rightarrow S^3 \) so that

\[
\hat{\Sigma}(F) = \hat{\Sigma}(F^T F).
\]

Significance:
The \( \hat{\Sigma} \) only depends on \( F^T F \).
Elasticity Theory

Axiom of Material Frame-Indifference

Given physical invariances, we make the assumption that the Cauchy stress vector \( \mathbf{t}(x, n) = T(x)n \) is independent of the choice of coordinates in the sense

\[
\mathbf{Q}^\mathbf{t}(\mathbf{Q}F) = \mathbf{Q}^\mathbf{t}(F) \mathbf{Q}^T,
\]

\[\forall \mathbf{Q} \in O_3^+.\]

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Theorem

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation \( \mathbf{Q} \in O_3^+ \) that

\[
\hat{\mathbf{T}}(\mathbf{Q}F) = \mathbf{Q}^T \hat{\mathbf{T}}(F) \mathbf{Q}.
\]

We also have there exists a mapping \( \hat{\mathbf{\Sigma}} : S_3 \rightarrow S_3 \) so that

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Axiom of Material Frame-Indifference

Given physical invariances, we make the assumption that the Cauchy stress vector $t(x, n) = T(x)n$ is independent of the choice of coordinates in the sense

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The Axiom of Material Frame-Indifference

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Elasticity Theory

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Elasticity Theory

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We also have there exists a mapping \( \hat{\Sigma} : S^3_+ \to S^3 \) so that

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\hat{\Sigma}(F) = \hat{\Sigma}(F^TF).
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Elasticity Theory

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\]

We also have there exists a mapping \( \hat{\Sigma} : S_3^3 \to S_3^3 \) so that

\[
\hat{\Sigma}(F) = \hat{\Sigma}(F^TF).
\]

**Significance:** The \( \hat{\Sigma} \) only depends on \( F^TF \).

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Axiom of Material Frame-Indifference

Given physical invariances, we make the assumption that the Cauchy stress vector $\mathbf{t}(\mathbf{x}, \mathbf{n}) = T(\mathbf{x})\mathbf{n}$ is independent of the choice of coordinates in the sense

$$Q \mathbf{t}(\mathbf{x}, \mathbf{n}) = \mathbf{t}(Q\mathbf{x}, Q\mathbf{n}), \quad \forall Q \in O_3^+.$$ 

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When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O_3^+$ that

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Theorem

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O^3$ that

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$$\hat{\Sigma}(F) = \tilde{\Sigma}(F^TF).$$

Proof:
Elasticity Theory

Theorem

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation \( Q \in O^+_3 \) that

\[
\hat{T}(QF) = Q \hat{T}(F) Q^T.
\]

We also have there exists a mapping \( \hat{\Sigma} : S^3_+ \rightarrow S^3 \) so that

\[
\hat{\Sigma}(F) = \check{\Sigma}(F^T F).
\]

Proof:

This follows by rotating the deformed body to obtain the relations
Theorem

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We also have there exists a mapping $\hat{\Sigma} : S^3_+ \rightarrow S^3$ so that

$$\hat{\Sigma}(F) = \tilde{\Sigma}(F^T F).$$

Proof:
This follows by rotating the deformed body to obtain the relations

$$x \mapsto Qx, \ \phi \mapsto Q\phi, \ \nabla \phi \mapsto Q\nabla \phi, \ n \mapsto Q^{-T}n = Qn, \ t(x, n) \mapsto Qt(x, n).$$
Elasticity Theory

**Theorem**

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O^3$ that

$$\hat{T}(QF) = Q\hat{T}(F)Q^T.$$  

We also have there exists a mapping $\hat{\Sigma} : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ so that

$$\hat{\Sigma}(F) = \tilde{\Sigma}(F^TF).$$

**Proof:**

This follows by rotating the deformed body to obtain the relations

$$\mathbf{x} \mapsto Q\mathbf{x}, \; \phi \mapsto Q\phi, \; \nabla \phi \mapsto Q\nabla \phi, \; \mathbf{n} \mapsto Q^{-T}\mathbf{n} = Q\mathbf{n}, \; t(\mathbf{x}, \mathbf{n}) \mapsto Qt(\mathbf{x}, \mathbf{n}).$$

From frame-indifference axiom, we have $t(Q\mathbf{x}, Q\mathbf{n}) \mapsto Qt(\mathbf{x}, \mathbf{n})$ and $\hat{T}(QF)Q \cdot \mathbf{n} = Q\hat{T}(F) \cdot \mathbf{n}$, using $Q^TQ = I$. 

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Theorem

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O_3^+$ that

$$\hat{T}(QF) = Q \hat{T}(F)Q^T.$$  

We also have there exists a mapping $\hat{\Sigma} : S^3_{>0} \rightarrow S^3$ so that

$$\hat{\Sigma}(F) = \tilde{\Sigma}(F^T F).$$

**Proof:**

This follows by rotating the deformed body to obtain the relations

$$x \mapsto Qx, \quad \phi \mapsto Q\phi, \quad \nabla\phi \mapsto Q\nabla\phi, \quad n \mapsto Q^{-T}n = Qn, \quad t(x, n) \mapsto Qt(x, n).$$

From frame-indifference axiom, we have $t(Qx, Qn) \mapsto Qt(x, n)$ and $\hat{T}(QF)Q \cdot n = Q \hat{T}(F) \cdot n$, using $Q^TQ = I$.

This implies $\hat{T}(QF) \cdot n = Q \hat{T}(F)Q^T \cdot n, \quad \forall n \in S^2 \Rightarrow \hat{T}(QF) = Q \hat{T}(F)Q^T.$
Elasticity Theory

**Theorem**

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation \( Q \in O_3^+ \) that

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We also have there exists a mapping \( \hat{\Sigma} : \mathbb{S}_+^3 \rightarrow \mathbb{S}^3 \) so that

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\hat{\Sigma}(F) = \hat{\Sigma}(F^T F).
\]

**Proof:**

This follows by rotating the deformed body to obtain the relations

\[
\mathbf{x} \mapsto Q\mathbf{x}, \quad \phi \mapsto Q\phi, \quad \nabla\phi \mapsto Q\nabla\phi, \quad \mathbf{n} \mapsto Q^{-T}\mathbf{n} = Q\mathbf{n}, \quad \mathbf{t}(\mathbf{x}, \mathbf{n}) \mapsto Q\mathbf{t}(\mathbf{x}, \mathbf{n}).
\]

From frame-indifference axiom, we have \( \mathbf{t}(Q\mathbf{x}, Q\mathbf{n}) \mapsto Q\mathbf{t}(\mathbf{x}, \mathbf{n}) \) and \( \hat{T}(QF)Q \cdot \mathbf{n} = Q \hat{T}(F) \cdot \mathbf{n} \), using \( Q^T Q = I \). This implies \( \hat{T}(QF) \cdot \mathbf{n} = Q \hat{T}(F)Q^T \cdot \mathbf{n}, \quad \forall \mathbf{n} \in S^2 \Rightarrow \hat{T}(QF) = Q \hat{T}(F) Q^T. \)

From \( \hat{\Sigma}(F) := \det(F)F^{-1} \hat{T}(F)F^{-T} \), we have \( \hat{\Sigma}(QF) = \hat{\Sigma}(F), \quad \forall Q \in O_3^+ \).
Theorem

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O^3_+$ that

$$\hat{T}(QF) = Q\hat{T}(F)Q^T.$$ 

We also have there exists a mapping $\hat{\Sigma} : S^3_+ \to S^3$ so that

$$\hat{\Sigma}(F) = \hat{\Sigma}(F^TF).$$

Proof:
This follows by rotating the deformed body to obtain the relations

$$x \mapsto Qx, \quad \phi \mapsto Q\phi, \quad \nabla\phi \mapsto Q\nabla\phi, \quad n \mapsto Q^{-T}n = Qn, \quad t(x, n) \mapsto Qt(x, n).$$

From frame-indifference axiom, we have $t(Qx, Qn) \mapsto Qt(x, n)$ and $\hat{T}(QF)Q \cdot n = \hat{T}(F) \cdot n$, using $Q^TQ = I$.

This implies $\hat{T}(QF) \cdot n = Q\hat{T}(F)Q^T \cdot n$, $\forall n \in S^2 \Rightarrow \hat{T}(QF) = Q\hat{T}(F)Q^T$.

From $\hat{\Sigma}(F) := \det(F)F^{-1}\hat{T}(F)F^{-T}$, we have $\hat{\Sigma}(QF) = \hat{\Sigma}(F)$, $\forall Q \in O^3_+$.

Now consider product $F^TF = G^TG$ for any $F$ and $G$ invertible. Let $Q = GF^{-1}$, then $Q^TQ = I$, $\det(Q) > 0$. 

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**Theorem**

When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation $Q \in O_3^+$ that

$$\hat{T}(QF) = Q \hat{T}(F)Q^T.$$  

We also have there exists a mapping $\hat{\Sigma} : S_3^+ \to S_3^+$ so that

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**Proof:**

This follows by rotating the deformed body to obtain the relations

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From frame-indifference axiom, we have $t(Qx, Qn) \mapsto Qt(x, n)$ and $\hat{T}(QF)Q \cdot n = Q \hat{T}(F) \cdot n$, using $Q^T Q = I$.

This implies $\hat{T}(QF) \cdot n = Q \hat{T}(F)Q^T \cdot n, \forall n \in S^2 \Rightarrow \hat{T}(QF) = Q \hat{T}(F)Q^T$.

From $\hat{\Sigma}(F) := \det(F)F^{-1} \hat{T}(F)F^{-T}$, we have $\hat{\Sigma}(QF) = \hat{\Sigma}(F), \forall Q \in O_3^+$.

Now consider product $F^T F = G^T G$ for any $F$ and $G$ invertible. Let $Q = GF^{-1}$, then $Q^T Q = I$, $\det(Q) > 0$.

This gives $\hat{\Sigma}(F) = \hat{\Sigma}(QF) = \hat{\Sigma}(G)$ so that $\hat{\Sigma}$ only depends on the product $F^T F$.  

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Theorem
When the Axiom of Material Frame-Indifference holds, we have for every orthogonal transformation \( Q \in O^3_+ \) that
\[
\hat{T}(QF) = Q \hat{T}(F)Q^T.
\]
We also have there exists a mapping \( \hat{\Sigma} : S^3_+ \rightarrow S^3 \) so that
\[
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\]

Proof:
This follows by rotating the deformed body to obtain the relations
\[
x \mapsto Qx, \ \phi \mapsto Q \phi, \ \nabla \phi \mapsto Q \nabla \phi, \ n \mapsto Q^{-T}n = Qn, \ t(x,n) \mapsto Qt(x,n).
\]
From frame-indifference axiom, we have \( t(Qx, Qn) \mapsto Qt(x,n) \) and \( \hat{T}(QF)Q \cdot n = Q \hat{T}(F)Q^T \cdot n \), using \( Q^T Q = I \).
This implies \( \hat{T}(QF) \cdot n = Q \hat{T}(F)Q^T \cdot n \), \( \forall n \in S^2 \) \( \Rightarrow \hat{T}(QF) = Q \hat{T}(F)Q^T \).
From \( \hat{\Sigma}(F) := \det(F)F^{-1} \hat{T}(F)F^{-T} \), we have \( \hat{\Sigma}(QF) = \hat{\Sigma}(F) \), \( \forall Q \in O^3_+ \).
Now consider product \( F^T F = G^T G \) for any \( F \) and \( G \) invertible. Let \( Q = GF^{-1} \), then \( Q^T Q = I \), \( \det(Q) > 0 \).
This gives \( \hat{\Sigma}(F) = \hat{\Sigma}(QF) = \hat{\Sigma}(G) \) so that \( \hat{\Sigma} \) only depends on the product \( F^T F \).
Isotropic Materials

A material is isotropic if
$$\hat{T}(F) = \hat{T}(FQ), \forall Q \in O_3^+.$$ This is equivalent to
$$\hat{T}(F) = \hat{T}(FF^T).$$ Significance: Isotropic materials have the same properties in all directions remaining the same when rotating the reference body. Note the order $FQ$ is important (not same as $QF$).

Invariants: The material responses depend only on invariants of the matrix $A = FF^T$ (also of $A^T = F^TF$).

We define the triple invariants $\iota_A = (\iota_1(A), \iota_2(A), \iota_3(A))$ as coefficients of
$$\det(\lambda I - A) = \lambda^3 - \iota_1(A) \lambda^2 + \iota_2(A) \lambda - \iota_3(A).$$ Invariants can be expressed as
$$\iota_1(A) := \text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3,$$
$$\iota_2(A) := \frac{1}{2}(\text{trace}(A)^2 - \text{trace}(A^2)) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3,$$
$$\iota_3(A) := \det(A) = \lambda_1 \lambda_2 \lambda_3.$$ Provides convenient way to model many isotropic materials.
Elasticity Theory

Isotropic Materials

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A material is \textit{isotropic} if

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Elasticity Theory

Isotropic Materials

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Provides convenient way to model many isotropic materials.

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This is equivalent to

\[ \hat{T}(F) = \hat{T}(FF^T). \]

**Significance:** Isotropic materials have the same properties in all directions remaining the same when rotating the reference body. Note the order \( FQ \) is important (not same as \( QF \)).
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Elasticity Theory

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Provides convenient way to model many isotropic materials.
The response function $\hat{T}^{3} \to S^{3}$ is objective and isotropic if and only if it has the form $\hat{T}(F) = \bar{T}(FF^T)$ with $\bar{T} : S^3 \to S^3$.

The $\iota_B$ denotes the triple of invariants of $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Significance: Characterizes the conditions under which constitutive laws are frame-indifferent and isotropic.
Rivlin-Ericksen (RE) Theorem

The response function $\hat{T} : \mathbb{M}^3_+ \rightarrow \mathbb{S}^3$ is *objective* and *isotropic* if and only if it has the form $\hat{T}(F) = \bar{T}(FF^T)$ with

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Elasticity Theory
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The second Piola-Kirchhoff stress tensor $\Sigma$ is objective and isotropic iff

$$\Sigma(F) = \tilde{\Sigma}(\nabla \phi^T \nabla \phi)$$

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The $\gamma_i = \gamma_i(\iota C)$ are functions of the triple of invariants $\iota C$.

Proof: We use the Cayley-Hamilton formula for $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$B^3 - \iota_1 (B^2) B + \iota_2 (B) B - \iota_3 I = 0.$$ 

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By the CH formula, we can eliminate the $I$ to obtain

$$\bar{T}(B) = \tilde{\beta}_1 B + \tilde{\beta}_2 B^2 + \tilde{\beta}_3 B^3.$$ 

Multiply on left by $\det(F) F^{-1}$ and on right by $F^{-T}$ to reformulate as $\hat{\Sigma}$ with Cauchy-Green tensor $C = F^T F$.

We use invariance to choose frame with $FF^T = B$. ■
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$\blacksquare$
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Elasticity Theory

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Elasticity Theory

**Corollary to RE**

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\end{align*}
$$

The $\gamma_i = \gamma_i(t_C)$ are functions of the triple of invariants $t_C$.

**Proof:**

We use the Cayley-Hamilton formula for $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$
B^3 - \nu_1(B)B^2 + \nu_2(B)B - \nu_3(B)I = 0.
$$

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Multiply on left by $\det(F)F^{-1}$ and on right by $F^{-T}$ to reformulate as $\hat{\Sigma}$ with Cauchy-Green tensor $C = F^TF$.

We use invariance to choose frame with $FF^T = B$. ■
For an objective isotropic material the second Piola-Kirchhoff stress is of form \( \tilde{\Sigma}(C) = \gamma_0 I + \gamma_1 C + \gamma_2 C^2 \).

Suppose that \( \gamma_i \) are continuously differentiable functions of \( \iota_j(E) \), then there exists constants \( \pi, \lambda, \mu \) so that \( \tilde{\Sigma}(C) = \tilde{\Sigma}(I + 2E) = -\pi I + \lambda \text{trace}(E) I + 2\mu E + o(E) \), as \( E \to 0 \).

Proof (sketch): From definition of \( \iota_j(E) \) we have \( \iota_1(E) \) contributes in the expressions.

From \( C = I + 2E \), \( C^2 = I + 4E + o(E) \), to obtain leading order we expand as

\[
\gamma_0(E) = a_0 + b_1 \iota_1(E) + o(E),
\gamma_1(E) = a_1 + O(E),
\gamma_2(E) = a_2 + O(E).
\]

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Using \( \iota_1(E) = \text{trace}(E) \) the result follows.

Significance: Gives general constitutive relation expressed in terms of strain \( E \) when deformations are small.

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Proof (sketch):

From definition of \( \iota_j(A) \) we have \( \iota_j(E) = O(E^i) \).
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Remark trace($\epsilon$) $\approx$ div($u$) for incompressibility. Lame’ constants: $\lambda$ change in density and $\mu$ shear modulus.
For an objective isotropic material the second Piola-Kirchhoff stress is of form $\tilde{\Sigma}(C) = \gamma_0 I + \gamma_1 C + \gamma_2 C^2$. Suppose that $\gamma_i$ are continuously differentiable functions of $\nu_j(E)$, then there exists constants $\pi, \lambda, \mu$ so that

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\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad \text{Poisson ratio}
\]

Paul J. Atzberger, UCSB

Finite Element Methods

http://atzberger.org/
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From considerations in the physics, we have \( \lambda > 0, \mu > 0 \) and \( E > 0, 0 < \nu < \frac{1}{2} \).
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From considerations in the physics, we have \( \lambda > 0, \mu > 0 \) and \( E > 0, 0 < \nu < \frac{1}{2} \).

**Remark:** For small deformations, if we replace linearization in \( E \) with linearization in \( \epsilon \) approach is called geometrically linear theory.
Hyperelastic Materials

Definition

A hyperelastic material is characterized by the existence of an energy functional $\hat{W}: \Omega \times M_3^+ \rightarrow \mathbb{R}$ so that

$$\hat{T}(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F), \quad \forall x \in \Omega, F \in M_3^+.$$ 

Equilibrium state of an elastic body:

$$-\text{div} \hat{T}(x, \nabla \varphi(x)) = f(x), \quad x \in \Omega$$

$$\hat{T}(x, \nabla \varphi(x)) = g(x), \quad x \in \Gamma_1$$

$$\varphi(x) = \varphi(x_0), \quad x \in \Gamma_0.$$ 

Variational principle:

If $f = \text{grad} F$, $g = \text{grad} G$, we have a variational principle with the functional

$$I[\psi] = \int_{\Omega} (\hat{W}(x, \nabla \psi(x)) - F(\psi(x))) \, dx + \int_{\Gamma_1} G(\psi(x)) \, dx.$$ 

We require that $\psi$ satisfies the boundary conditions on $\Gamma_1$, $\Gamma_0$ and local injectivity $\det(\nabla \psi(x)) > 0$. 

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A hyperelastic materials is characterized by the existence of an energy functional $\hat{W} : \Omega \times M_3^+ \to \mathbb{R}$ so that
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Objective Material: The $\hat{W}(x, \cdot)$ is function only of Cauchy-Green Tensor $C = F^T F$ as
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Isotropic Materials:
**Hyperelastic Materials**

**Definition**

A **hyperelastic materials** is characterized by the existence of an energy functional $\hat{W} : \Omega \times \mathbb{M}_3^+ \rightarrow \mathbb{R}$ so that

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**Objective Material:** The $\hat{W}(x, \cdot)$ is function only of Cauchy-Green Tensor $C = F^T F$ as

$$\hat{W}(x, F) = \tilde{W}(x, F^T F), \ \tilde{\Sigma}(x, C) = 2 \frac{\partial \tilde{W}(x, C)}{\partial C}, \ \forall C \in S^3_+. \tag{2}$$

**Isotropic Materials:**

$$\hat{W}(x, F) = \tilde{W}(x, FQ), \ \forall F \in \mathbb{M}^3_+, \ Q \in O_3^+. \tag{3}$$
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**Isotropic Materials:**

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\]

**Isotropic Materials (small deformations):**
Hyperelastic Materials

Definition

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\tilde{W}(x, C) = \frac{\lambda}{2} (\text{trace}E)^2 + \mu E : E + o(E^2),
\]
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A *hyperelastic materials* is characterized by the existence of an energy functional $\hat{W} : \Omega \times \mathbb{M}_+^3 \to \mathbb{R}$ so that

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where $C = I + 2E$, $A : B = \sum_{ij} A_{ij} B_{ij} = \text{trace}(A^T B)$.
A hyperelastic materials is characterized by the existence of an energy functional $\hat{W} : \Omega \times \mathbb{M}_+^3 \rightarrow \mathbb{R}$ so that

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**St. Venant-Kirchhoff Materials:**

$\hat{\mathcal{W}}(x, F) = \lambda \left( \text{trace} \ F \right)^2 + \mu F : F = \lambda \left( \text{trace} \ F \right)^2 + \mu \text{trace} C.$

$\hat{\mathcal{W}}(x, C) = \frac{1}{2} \mu (\text{trace}(C - I) + 2 \beta (\text{det} C - \beta)/\beta - 1), \text{where} \ \beta = \frac{2 \nu}{1 - 2 \nu}.$
Hyperelastic Materials

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\tilde{W}(x, F) = \frac{\lambda}{2} (\text{trace} F)^2 + \mu F : F = \frac{\lambda}{2} (\text{trace} F)^2 + \mu \text{trace} C.
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Neo-Hookean Materials:
Hyperelastic Materials

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\]

St. Venant-Kirchhoff Materials:

\[
\tilde{W}(x, F) = \frac{\lambda}{2} \left( \text{trace}F \right)^2 + \mu F : F = \frac{\lambda}{2} \left( \text{trace}F \right)^2 + \mu \text{trace}C.
\]

Neo-Hookean Materials:

\[
\tilde{W}(x, C) = \frac{1}{2} \mu \left( \text{trace}(C - I) + \frac{2}{\beta} \left( (\det C)^{-\beta/2} - 1 \right) \right),
\]
Hyperelastic Materials

Definition

A hyperelastic materials is characterized by the existence of an energy functional $\hat{W} : \Omega \times M^3_+ \rightarrow \mathbb{R}$ so that

$$\hat{T}(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F), \quad \forall x \in \Omega, F \in M^3_+. $$

St. Venant-Kirchhoff Materials:

$$\tilde{W}(x, F) = \frac{\lambda}{2} (\text{trace} F)^2 + \mu F : F = \frac{\lambda}{2} (\text{trace} F)^2 + \mu \text{trace} C.$$  

Neo-Hookean Materials:

$$\tilde{W}(x, C) = \frac{1}{2} \mu (\text{trace}(C - I)) + \frac{2}{\beta} \left((\text{det}\ C)^{-\beta/2} - 1\right),$$

where $\beta = 2\nu / 1 - 2\nu$.  

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Linear Elasticity Theory

Assumptions:
Will restrict to case of small deformations for linearized isotropic materials.
Do not have to distinguish between stress tensors in this case.

Notation:
We use $\sigma$ instead of $\Sigma$ and $\epsilon$ instead of $E$.

Variational Problem
\[ \Pi := \int_{\Omega} \left( \frac{1}{2} \epsilon: \sigma - f \cdot u \right) \, dV + \int_{\Gamma} g \cdot u \, dA. \]
The tensor product $\epsilon: \sigma = \epsilon_{ij} \sigma_{ij}$. Note, the $\sigma, \epsilon, u$ are not independent here.

Kinematic Equations
The strain and displacement are related by
\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \epsilon = \epsilon(u) = \nabla(s) \cdot u. \]
The stress is related by the constitutive relation
\[ \epsilon = 1 + \nu E \sigma - \nu E \text{trace}(\sigma) I. \]
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\Pi := \int_\Omega \left( \frac{1}{2} \epsilon : \sigma - f \cdot u \right) dV + \int_{\Gamma_1} g \cdot u dA,
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Aim: We would like to formulate the mechanics in terms of the hyperelastic theory and equilibrium conditions given by \[ I[\psi]. \]

Need to specify \( \hat{W}, F, G. \)

We use that \( \text{trace}(\epsilon) = \frac{(1 - 2\nu)}{E} \text{trace}(\sigma), \) and solving for \( \sigma \) we have

\[ \sigma = E \left( 1 + \nu \right) \left( \epsilon + \nu \frac{1}{2} - 2\nu \frac{\text{trace}(\epsilon)}{I} \right). \]

We also can use that \( \epsilon : I = \text{trace}(\epsilon) \) so that

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Remark: This leads to a mixed formulation of weak problem.

Formulations: There are at least three distinct approaches in the literature:

(i) Displacement Formulation,
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Displacement Formulation

Variational Principle for Displacement Formulation

\[ \Pi[v] = \int_{\Omega} (\mu \epsilon[v] : \epsilon[v] + \lambda (\text{div}(v))^2 - f \cdot v) \, dV + \int_{\Gamma_1} g \cdot v \, dA \rightarrow \min \]

This is obtained by eliminating \( \sigma \) using \( \sigma = C \epsilon \), where

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix} = E(1 + \nu)(1 - 2\nu) \begin{bmatrix}
1 & -\nu & -\nu & -\nu & 0 & -\nu \\
-\nu & 1 & -\nu & 0 & 1 & -\nu \\
-\nu & -\nu & 1 & 1 - 2\nu & 0 & 1 - 2\nu \\
-\nu & 0 & 1 - 2\nu & 1 & 1 - 2\nu & 0 \\
0 & 1 & -\nu & -\nu & 1 & -\nu \\
-\nu & -\nu & 1 - 2\nu & 0 & -\nu & 1
\end{bmatrix} \begin{bmatrix}
\epsilon_{11} \\
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\epsilon_{13} \\
\epsilon_{23}
\end{bmatrix}.
\]

The variational principle above is obtained from this with notation \( \epsilon = \nabla(s) v \) and

\[
\Pi[v] = \int_{\Omega} \left( \frac{1}{2} \nabla(s) v : C \nabla(s) v - f \cdot v \right) \, dV + \int_{\Gamma_1} g \cdot v \, dA.
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Displacement Formulation

Variational Principle for Displacement Formulation

\[ \Pi[v] = \int_{\Omega} \left( \mu \epsilon[v] : \epsilon[v] + \frac{\lambda}{2} (\text{div}(v))^2 - f \cdot v \right) dV_x + \int_{\Gamma_1} g \cdot v \, dA_x \rightarrow \text{min} \]
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\begin{bmatrix}
1 - \nu & \nu & \nu \\
\nu & 1 - \nu & \nu \\
\nu & \nu & 1 - \nu \\
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0 & 1 - 2\nu & 1 - 2\nu \\
\end{bmatrix}
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Variational Principle for Displacement Formulation

\[ \Pi[v] = \int_{\Omega} \left( \mu \varepsilon[v] : \varepsilon[v] + \frac{\lambda}{2} (\text{div}(\nu))^2 - f \cdot \nu \right) \text{d}V_x + \int_{\Gamma_1} g \cdot \nu \text{d}A_x \rightarrow \min \]

This is obtained by eliminating \( \sigma \) using \( \sigma = C\varepsilon \), where

\[ \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \\ 0 & 1-2\nu & 0 \\ 0 & 1-2\nu & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}. \]

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The variational principle above is obtained from this with notation \( \epsilon = \nabla^{(s)} v \) and

\[ \Pi[v] = \int_{\Omega} \left( \frac{1}{2} \nabla^{(s)} v : C \nabla^{(s)} v - f \cdot v \right) dV_x + \int_{\Gamma_1} g \cdot v dA_x. \]
Displacement Formulation

Variational Principle for Displacement Formulation

$$
\Pi[v] = \int_{\Omega} \left( \mu \varepsilon[v] : \varepsilon[v] + \frac{\lambda}{2} (\text{div}(v))^2 - f \cdot v \right) dV + \int_{\Gamma_1} g \cdot v dA \rightarrow \min
$$

St. Venant-Kirchhoff Materials:

The weak formulation specializes to

$$
2 \mu (\nabla (s) u, \nabla (s) v)_{0} + \lambda (\text{div } u, \text{div } v)_{0} = (f, v)_{0} - (g, v)_{\Gamma_1}, \forall v \in H^1 \Gamma
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\[ H^1_I := \{ v \in H^1(\Omega)^3 : v(x) = 0, \forall x \in \Gamma_0 \} \]
Variational Principle for Displacement Formulation

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By introducing \( L - 2 \) inner-product notation, we can express as
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\[ \Pi[\nu] = \int_{\Omega} \left( \mu \epsilon[\nu] : \epsilon[\nu] + \frac{\lambda}{2} (\text{div}(\nu))^2 - f \cdot \nu \right) \, dV_x + \int_{\Gamma_1} g \cdot \nu \, dA_x \to \min \]

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Displacement Formulation

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**Strong Form Elliptic PDEs:** For the St. Venant-Kirchhoff we have

\[
\begin{align*}
-2\mu \text{div} \varepsilon(u) - \lambda \text{grad div} u &= f, \quad x \in \Omega, \\
\text{div} u &= 0, \quad x \in \Gamma_0, \\
\sigma(u) \cdot n &= g \quad x \in \Gamma_1.
\end{align*}
\]
Weak Formulation (Hellinger and Reissner):

\[ \begin{align*}
(C^{-1}\sigma - \nabla (s u), \tau)_{0} &= 0, \\
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\end{align*} \]

This is related to the Displacement Formulation by using solution \( u \) to define \( \sigma := C \nabla (s u) \in L^2(\Omega) \).

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Weak Formulation II:

We find it helpful later to organize the weak problem as

\[\begin{align*}
X &= L^2(\Omega), \\
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Hu and Washizu Mixed Method Formulation

Weak Formulation (Hu and Washizu):

\[(C \epsilon - \sigma, \eta)_0 = 0, \quad \forall \eta \in L^2(\Omega),\]

\[(\epsilon - \nabla (s) u, \tau)_0 = 0, \quad \forall \tau \in L^2(\Omega),\]

\[-(\sigma, \nabla (s) v)_0 = -(f, v)_0 + \int_{\Gamma_1} g \cdot v \, dx, \quad \forall v \in H^1_{\Gamma}(\Omega).\]

Weak Formulation II:

We find it helpful later to organize the weak problem as

\[X = L^2(\Omega) \times L^2(\Omega),\]

\[M = H^1_{\Gamma}(\Omega),\]

\[a(\epsilon, \sigma, \eta, \tau) = (C \epsilon, \eta)_0,\]

\[b(\eta, \tau, v) = (\tau, \nabla (s) v - \epsilon)_0.\]

Allow typically more accurate calculation of stresses since represented directly as degrees of freedom.

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\begin{align*}
X &= L_2(\Omega) \times L_2(\Omega), \quad M = H^1_G(\Omega) \\
a(\varepsilon, \sigma, \eta, \tau) &= (C, \varepsilon, \eta)_0, \quad b(\eta, \tau, \nu) = (\tau, \nabla^s\nu - \varepsilon)_0.
\end{align*}
\]

Allow typically more accurate calculation of stresses since represented directly as degrees of freedom.
Korn’s First Inequality

For $\Omega$ an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary, there exists a number $c = c(\Omega) > 0$ so that

$$\int_{\Omega} \varepsilon(v) : \varepsilon(v) \, dx + \|v\|_0^2 \geq c \|v\|_1^2, \quad \forall v \in H^1_\Omega.$$

Korn’s Second Inequality

For $\Omega \subset \mathbb{R}^3$ be an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary and $\Gamma_0 \subset \partial \Omega$ have positive two-dimensional measure. Then there exists a positive number $c' = c'(\Omega, \Gamma_0)$ so that

$$\int_{\Omega} \varepsilon(v) : \varepsilon(v) \, dx \geq c' \|v\|_1^1, \quad \forall v \in H^1_{\Gamma_0}(\Omega).$$

Here, $H^1_{\Gamma_0}(\Omega)$ is the closure of $\{v \in C^\infty : v(x) = 0, \forall x \in \Gamma_0\}$ with respect to norm $\|\cdot\|_1$. Useful in establishing that variational problems involving strain are elliptic.
Korn’s First Inequality

For Ω an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary,

$$\int_{\Omega} \varepsilon(v) : \varepsilon(v) \, dx + \|v\|_{L^2}^2 \geq c(\varepsilon) \|v\|_{H^1(\Omega)}^2, \quad \forall v \in H^1(\Omega).$$

Here, $H^1(\Omega)$ is the space of functions whose first derivatives are square-integrable, and $H^1(\Omega)$ is the closure of the space of smooth functions vanishing on the boundary $\Gamma_0$ with respect to the norm $\|\cdot\|_{H^1(\Omega)}$.

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For $\Omega \subset \mathbb{R}^3$ be an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary and $\Gamma_0 \subset \partial \Omega$ have positive two-dimensional measure.

Then there exists a positive number $c' = c'(\Omega, \Gamma_0)$ so that

$$\int_{\Omega} \varepsilon(v) : \varepsilon(v) \, dx \geq c' \|v\|_{H^{1/2}(\Gamma_0)}^2, \quad \forall v \in H^{1/2}(\Gamma_0)(\Omega).$$

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Displacement Formulation

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Displacement Formulation

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For $\Omega$ an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary, there exists a number $c = c(\Omega) > 0$ so that

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Useful in establishing that variational problems involving strain are elliptic.
Existence Theorem (Displacement Formulation)

Let $\Omega \subset \mathbb{R}^3$ be a domain with piecewise smooth boundary, and $\Gamma_0$ has positive two-dimensional measure. Then the variational problem of linear elasticity has exactly one solution. This follows by establishing the coercivity condition for the bilinear form in the variational problem. For the weak displacement formulation this is done using the Korn Inequalities. The Lax-Milgram Theorem then gives the well-posedness of the variational problem. There are results establishing conditions for well-posedness for the other formulations. These typically involve analysis establishing the Babuska-Brezzi inf-sup conditions hold (discussed with mixed method theory).
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Locking Phenomena

Nearly Incompressible Materials

Mixed methods can have trouble approximating responses in some regimes of material properties. Consider a nearly incompressible material, which corresponds to Lamé constants with $\lambda \gg \mu$.

In the displacement formulation on $v \in H^1_\Gamma$, we have

$$a(u, v) := \lambda (\text{div} u, \text{div} v) + 2\mu (\epsilon(u), \epsilon(v)) \leq a(v, v) \leq C \|v\|_2^2,$$

with $\alpha \leq \mu$ and $C \geq \lambda + 2\mu$.

Recall, in Céa's Lemma we obtained bound with $C/\alpha$ which suggests large pre-factors in incompressible limit. In practice, results in errors in the solution much larger than the approximation error of the finite element space. This manifests typically with displacements much smaller than expected, referred to as locking effects.

In the nearly incompressible regime, referred to as volume locking or Poisson locking.
Nearly Incompressible Materials

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Normly Incompressible Materials

Mixed methods can have trouble approximating responses in some regimes of material properties. Consider a nearly incompressible material, which corresponds to Lamé’ constants with \( \lambda \gg \mu \).

In the displacement formulation on \( v \in H^1_\Gamma \), we have

\[
a(u, v) := \lambda (\text{div} u, \text{div} v)_0 + 2\mu (\epsilon(u), \epsilon(v))_0, \quad \rightarrow \alpha \|v\|_2^2 \leq a(v, v) \leq C \|v\|_2^2,
\]

with \( \alpha \leq \mu \) and \( C \geq \lambda + 2\mu \). Recall, in Céa’s Lemma we obtained bound with \( C/\alpha \) which suggests large pre-factors in incompressible limit. In practice, results in errors in the solution much larger than the approximation error of the finite element space. Manifests typically with displacements much smaller than expected, referred to as locking effects. In the nearly incompressible regime, referred to as volume locking or Poisson locking.
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Locking Phenomena

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Consider a nearly incompressible material, which corresponds to Lame’ constants with

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In the displacement formulation on \( v \in H^1 \), we have
Nearly Incompressible Materials

Mixed methods can have trouble approximating responses in some regimes of material properties. Consider a nearly \textbf{incompressible} material, which corresponds to Lame’ constants with

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In the displacement formulation on \( v \in H^1 \), we have

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In the displacement formulation on \( v \in H^1_1 \), we have

\[
a(u, v) := \lambda(\text{div } u, \text{div } v)_0 + 2\mu(\epsilon(u), \epsilon(v))_0, \quad \rightarrow \quad \alpha \|v\|_1^2 \leq a(v, v) \leq C \|v\|_1^2, \quad \text{with} \quad \alpha \leq \mu \quad \text{and} \quad C \geq \lambda + 2\mu.
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Recall, in Céa’s Lemma we obtained bound with \( C/\alpha \) which suggests large pre-factors in incompressible limit. In practice, results in errors in the solution much larger than the approximation error of the finite element space. Manifests typically with displacements much smaller than expected, referred to as **locking effects**.
Locking Phenomena

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Consider a nearly incompressible material, which corresponds to Lamé’ constants with

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In the displacement formulation on \( v \in H^1_T \), we have

\[ a(u, v) := \lambda (\text{div} \, u, \text{div} \, v)_0 + 2\mu (\varepsilon(u), \varepsilon(v))_0, \quad \rightarrow \quad \alpha \|v\|_1^2 \leq a(v, v) \leq C \|v\|_1^2, \quad \text{with} \quad \alpha \leq \mu \quad \text{and} \quad C \geq \lambda + 2\mu. \]

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In the displacement formulation on \( v \in H^1_f \), we have

\[ a(u, v) := \lambda (\text{div } u, \text{div } v)_0 + 2\mu (\varepsilon(u), \varepsilon(v))_0, \quad \rightarrow \quad \alpha \| v \|_1^2 \leq a(v, v) \leq C \| v \|_1^2, \quad \text{with} \quad \alpha \leq \mu \quad \text{and} \quad C \geq \lambda + 2\mu. \]

**Remedy:** One approach is to reformulate as a mixed method to obtain saddle-point problem. Let \( p := \lambda \text{div } u, \)
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\[
(\text{div} u, q)_0 - \lambda^{-1} (p, q)_0 = 0, \quad \forall q \in L^2(\Omega).
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Locking Phenomena

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Can be shown this gives a stable problem and well-defined in the limit \( \lambda \to \infty \).
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**Discretization:** Choose appropriate finite element spaces for the mixed method (discussed in other lecture).