Variational Formulation of Elliptic PDEs

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206D: Finite Element Methods
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Variational Formulation

Definition

A bilinear form $b(\cdot, \cdot)$ is a mapping $b: V \times V \to \mathbb{R}$ on a linear space $V$ so that the following holds:

1. $b$ is linear in both components, so $L_v[w] = b(v, w)$ and $L_w[v] = b(v, w)$ are both linear maps.

2. $b$ is symmetric so $b(v, w) = b(w, v)$.

An inner-product is a symmetric bilinear form with the additional properties

3. $b(v, v) \geq 0$, $\forall v \in V$.

4. $b(v, v) = 0$, $\iff v \equiv 0$.

Examples:

1. $V = \{ w \mid w(x) = \sum_{k=1}^{n} c_k \phi_k(x) \}$ where $u = \sum_{k=1}^{n} a_k \phi_k$, $v = \sum_{k=1}^{n} b_k \phi_k$.
   We define $b(u, v) = \sum_k w_k a_k b_k$.
   When $w_k > 0$ and $\phi_k$ are linearly independent this is an inner-product.

2. $V = \mathbb{R}^m$ and $b(x, y) = x \cdot y$ for $x, y \in \mathbb{R}^n$.

3. $V = W^2_k(\Omega)$ with $\Omega \subset \mathbb{R}^n$ with $(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$. 

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1. $\mathcal{V} = \{ w | w(x) = \sum_{k=1}^{n} c_k \phi_k(x) \}$ where $u = \sum_{k=1}^{n} a_k \phi_k$, $v = \sum_{k=1}^{n} b_k \phi_k$, we define $b(u, v) = \sum_{k} w_k a_k b_k$.
   - When $w_k > 0$ and $\phi_k$ are linearly independent this is an inner-product.
2. $\mathcal{V} = \mathbb{R}^m$ and $b(x, y) = x \cdot y$ for $x, y \in \mathbb{R}^n$.
3. $\mathcal{V} = W^{k, 2}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ with $(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$. 

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   2. \( \mathcal{V} = \mathbb{R}^m \) and \( b(x, y) = x \cdot y \) for \( x, y \in \mathbb{R}^n \).
   3. \( \mathcal{V} = W^{m \times 2}(\Omega) \) with \( \Omega \subset \mathbb{R}^n \) where \( \langle u, v \rangle_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)} \).
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A bilinear form \( a(\cdot,\cdot) \) is bounded if there exists \( C < \infty \) so that
\[
|a(v,w)| \leq C \|v\|_V \|w\|_V, \quad \forall v, w \in V.
\]

Since \( a \) is linear this is equivalent to being continuous.

A bilinear form \( a(\cdot,\cdot) \) is coercive on \( V \subset H \) if there exists an \( \alpha > 0 \) so that
\[
a(v,v) \geq \alpha \|v\|_H^2,
\]

Lemma

Consider \( V \subset H \) a linear subspace of a Hilbert space \( H \).

If \( a \) is continuous on \( H \) and coercive on \( V \) then the space \( (V,a(\cdot,\cdot)) \) is a Hilbert space.

Proof:

Since \( a(\cdot,\cdot) \) is coercive we have
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so \( a \) is an inner-product and \( \|v\|_E = \sqrt{a(v,v)} \) is a norm.

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**Lemma**

Consider \( V \subset \mathcal{H} \) a linear subspace of a Hilbert space \( \mathcal{H} \).

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Suppose \( \{v_k\} \) is a Cauchy sequence in \((V, \| \cdot \|_E)\).
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Suppose \( \{v_k\} \) is a Cauchy sequence in \((V, \| \cdot \|_E)\), then by coercivity \( \{v_k\} \) is also Cauchy in \((H, \| \cdot \|)\).
Variational Formulation

**Definition**

A bilinear form $a(\cdot, \cdot)$ is **bounded** if there exists $C < \infty$ so that

$$|a(v, w)| \leq C\|v\|_V\|w\|_V, \forall v, w \in V.$$  

Since $a$ is linear this is equivalent to being **continuous**.

A bilinear form $a(\cdot, \cdot)$ is **coercive** on $V \subset H$ if there exists an $\alpha$ so that

$$a(v, v) \geq \alpha\|v\|^2_H$$

**Lemma**

Consider $V \subset H$ a linear subspace of a Hilbert space $H$. If $a$ is continuous on $H$ and coercive on $V$ then the space $(V, a(\cdot, \cdot))$ is a Hilbert space.

**Proof (continued):**

Suppose $\{v_k\}$ is a Cauchy sequence in $(V, \|\cdot\|_E)$, then by coercivity $\{v_k\}$ is also Cauchy in $(H, \|\cdot\|)$. By completeness of $H$ there exists a $v \in H$ so $v_n \to v$ in $\|\cdot\|_H$. 

---

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Variational Formulation

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Variational Formulation

Definition

A bilinear form \( a(\cdot, \cdot) \) is **bounded** if there exists \( C < \infty \) so that

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\]

Since \( a \) is linear this is equivalent to being **continuous**.

A bilinear form \( a(\cdot, \cdot) \) is **coercive** on \( V \subset \mathcal{H} \) if there exists an \( \alpha \) so that

\[
a(v, v) \geq \alpha \|v\|_H^2
\]

Lemma

Consider \( V \subset \mathcal{H} \) a linear subspace of a Hilbert space \( \mathcal{H} \). If \( a \) is continuous on \( \mathcal{H} \) and coercive on \( V \) then the space \((V, a(\cdot, \cdot))\) is a Hilbert space.

Proof (continued):

Suppose \( \{v_k\} \) is a Cauchy sequence in \((V, \|\cdot\|_E)\), then by coercivity \( \{v_k\} \) is also Cauchy in \((\mathcal{H}, \|\cdot\|)\). By completeness of \( \mathcal{H} \) there exists a \( v \in \mathcal{H} \) so \( v_n \to v \) in \( \|\cdot\|_H \). Since \( V \) is closed in \( \mathcal{H} \) by def. of a subspace we have \( v \in V \). Now \( \|v - v_k\|_E \leq c \|v - v_k\|_H \) since \( a \) is bounded,
Variational Formulation

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A bilinear form $a(\cdot, \cdot)$ is **bounded** if there exists $C < \infty$ so that

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Suppose $\{v_k\}$ is a Cauchy sequence in $(V, \|\cdot\|_E)$, then by coercivity $\{v_k\}$ is also Cauchy in $(H, \|\cdot\|)$. By completeness of $H$ there exists a $v \in H$ so $v_n \to v$ in $\|\cdot\|_H$. Since $V$ is closed in $H$ by def. of a subspace we have $v \in V$. Now $\|v - v_k\|_E \leq c \|v - v_k\|_H$ since $a$ is bounded, so $v_k$ converges to $v$ in $\|\cdot\|_E$ showing $V$ is complete.
**Variational Formulation**

**Definition**

A bilinear form \(a(\cdot, \cdot)\) is **bounded** if there exists \(C < \infty\) so that

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|a(v, w)| \leq C \|v\|_V \|w\|_V, \quad \forall v, w \in V.
\]

Since \(a\) is linear this is equivalent to being **continuous**.

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**Lemma**

Consider \(V \subset H\) a linear subspace of a Hilbert space \(H\). If \(a\) is continuous on \(H\) and coercive on \(V\) then the space \((V, a(\cdot, \cdot))\) is a Hilbert space.

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Suppose \(\{v_k\}\) is a Cauchy sequence in \((V, \|\cdot\|_E)\), then by coercivity \(\{v_k\}\) is also Cauchy in \((H, \|\cdot\|)\). By completeness of \(H\) there exists a \(v \in H\) so \(v_n \to v\) in \(\|\cdot\|_H\). Since \(V\) is closed in \(H\) by def. of a subspace we have \(v \in V\). Now \(\|v - v_k\|_E \leq c\|v - v_k\|_H\) since \(a\) is bounded, so \(v_k\) converges to \(v\) in \(\|\cdot\|_E\) showing \(V\) is complete. □
Variational Formulation

Definition

A symmetric variational problem satisfies the following:

i. Given \( F \in V' \), find \( u \) satisfying
\[
(a(u, v)) = F[v], \quad \forall v \in V,
\]
where

ii. \((H, (\cdot, \cdot))\) is a Hilbert space,

iii. \( V \) is a subspace of \( H \),

iv. \( a(\cdot, \cdot) \) is a symmetric bilinear form that is bounded on \( H \) and coercive on \( V \).

Theorem

For the variational problem \((\ast)\), if the conditions ii-iv hold then there exists a unique solution \( u \in V \) solving \((\ast)\).
Definition

A symmetric variational problem satisfies the following

1. Given $F \in V'$, find $u$ satisfying
   \[ a(u, v) = F(v), \quad \forall v \in V, \]
   \[ (*) \]
2. $(H, (\cdot, \cdot))$ is a Hilbert space,
3. $V$ is a subspace of $H$,
4. $a(\cdot, \cdot)$ is a symmetric bilinear form that is bounded on $H$ and coercive on $V$.

Theorem

For the variational problem $(*)$, if the conditions ii-iv hold then there exists a unique solution $u \in V$ solving $(*)$. 

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A symmetric variational problem satisfies the following

i. Given $F \in \mathcal{V}'$, find $u$ satisfying

$$a(u, v) = F[v], \quad \forall v \in \mathcal{V}, \quad (*)$$

where
Variational Formulation

Definition

A **symmetric variational problem** satisfies the following:

1. Given $F \in \mathcal{V}'$, find $u$ satisfying
   \[ a(u, v) = F[v], \quad \forall v \in \mathcal{V}, \quad (\ast) \]
   where

2. $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space,
Variational Formulation

Definition

A **symmetric variational problem** satisfies the following

i. Given $F \in H'$, find $u$ satisfying

$$a(u, v) = F[v], \quad \forall v \in H,$$  \hfill (\star)$$

where

ii. $(H, (\cdot, \cdot))$ is a Hilbert space,

iii. $V$ is a subspace of $H$,
Variational Formulation

**Definition**

A *symmetric variational problem* satisfies the following

1. Given \( F \in \mathcal{V}' \), find \( u \) satisfying
   \[
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   \]
   where

2. \((\mathcal{H}, (\cdot, \cdot))\) is a Hilbert space,

3. \( \mathcal{V} \) is a subspace of \( \mathcal{H} \),

4. \( a(\cdot, \cdot) \) is a symmetric bilinear form that is bounded on \( \mathcal{H} \) and coercive on \( \mathcal{V} \).
Variational Formulation

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A **symmetric variational problem** satisfies the following

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Theorem

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Variational Formulation

Definition

A **symmetric variational problem** satisfies the following

i. Given $F \in V'$, find $u$ satisfying

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For the variational problem $(\ast)$, if the conditions ii-iv hold then there exists a unique solution $u \in V$ solving $(\ast)$.
Symmetric Variational Problem

Given $F \in \mathcal{V}'$, find $u$ satisfying
\[ a(u, v) = F[v], \quad \forall v \in \mathcal{V} \] (\*)

ii. $(\mathcal{H}, (\cdot, \cdot))$ Hilbert space, iii. $\mathcal{V}$ is a subspace of $\mathcal{H}$,
iv. $a$ symmetric, bounded on $\mathcal{H}$, coercive on $\mathcal{V}$.

Theorem

For the variational problem (\*), if the conditions ii-iv hold then there exists a unique solution $u \in \mathcal{V}$ solving (\*).

Proof:
Variational Formulation

Symmetric Variational Problem

Given \( F \in \mathcal{V}' \), find \( u \) satisfying
\[
a(u, v) = F[v], \quad \forall v \in \mathcal{V} \quad (\ast)
\]

ii. \((\mathcal{H}, (\cdot, \cdot))\) Hilbert space, iii. \( \mathcal{V} \) is a subspace of \( \mathcal{H} \),
iv. \( a \) symmetric, bounded on \( \mathcal{H} \), coercive on \( \mathcal{V} \).

Theorem

For the variational problem \((\ast)\), if the conditions ii-iv hold then there exists a unique solution \( u \in \mathcal{V} \) solving \((\ast)\).

Proof:
The conditions ensure that \( a(\cdot, \cdot) \) is an inner-product on \( \mathcal{V} \) and that \((\mathcal{V}, a(\cdot, \cdot))\) is a Hilbert space.
Variational Formulation

Symmetric Variational Problem

Given $F \in V'$, find $u$ satisfying

$$a(u, v) = F[v], \; \forall v \in V \quad (*)$$

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Theorem

For the variational problem $(*)$, if the conditions ii-iv hold then there exists a unique solution $u \in V$ solving $(*)$.

Proof:
The conditions ensure that $a(\cdot, \cdot)$ is an inner-product on $V$ and that $(V, a(\cdot, \cdot))$ is a Hilbert space. By Riesz Representation Theorem, all bounded linear functionals have representative $u$ in the inner-product.
Symmetric Variational Problem

Given $F \in \mathcal{V}'$, find $u$ satisfying
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This implies there exists $u$ satisfying $(*)$. 
Variational Formulation

Symmetric Variational Problem
Given $F \in V'$, find $u$ satisfying
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By Riesz Representation Theorem, all bounded linear functionals have representative $u$ in the inner-product.
This implies there exists $u$ satisfying $(\ast)$. ■
The Ritz-Galerkin Approximation is based on a finite-dimensional subspace $V_h \subset V$ and $F \in V'$. The problem is to find $u_h \in V_h$ so that

$$a(u_h, v) = F[v], \quad \forall v \in V_h,$$

Theorem

For the Ritz-Galerkin approximation problem $(\ast \ast)$, if the conditions ii-iv hold then there exists a unique solution $u_h \in V_h$ solving $(\ast \ast)$.

Proof:

This follows since $(V_h, a(\cdot, \cdot))$ is also a Hilbert space and we can again invoke the Riesz Representation Theorem to obtain representative $u_h$ that satisfies $(\ast \ast)$.

$\blacksquare$
Variational Formulation

**Definition**

The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace \( \mathcal{V}_h \subset \mathcal{V} \) and \( F \in \mathcal{V}' \). The problem is to find \( u_h \in \mathcal{V}_h \) so that

\[
\text{a}(u_h, v) = F[v], \quad \forall v \in \mathcal{V}_h,
\]

\[\text{(**)\quad \text{Theorem}}\]

For the Ritz-Galerkin approximation problem (**), if the conditions ii-iv hold then there exists a unique solution \( u_h \in \mathcal{V}_h \) solving (**).

**Proof:**

This follows since \((\mathcal{V}_h, \text{a}(\cdot, \cdot))\) is also a Hilbert space and we can again invoke the Riesz Representation Theorem to obtain representative \( u_h \) that satisfies (**).
The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ and $F \in \mathcal{V}'$. The problem is to find $u_h \in \mathcal{V}_h$ so that
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Definition

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Theorem

For the Ritz-Galerkin approximation problem (**), if the conditions ii-iv hold then there exists a unique solution $u_h \in \mathcal{V}_h$ solving (**).
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The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$ and $F \in \mathcal{V}'$. The problem is to find $u_h \in \mathcal{V}_h$ so that

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Theorem

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Variational Formulation

Definition

The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace \( \mathcal{V}_h \subset \mathcal{V} \) and \( F \in \mathcal{V}' \). The problem is to find \( u_h \in \mathcal{V}_h \) so that

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\]

\( (**) \)

Theorem

For the Ritz-Galerkin approximation problem (**)\( , \) if the conditions ii-iv hold then there exists a unique solution \( u_h \in \mathcal{V}_h \) solving (**).
**Definition**

The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace $V_h \subset V$ and $F \in V'$. The problem is to find $u_h \in V_h$ so that

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**Theorem**

For the Ritz-Galerkin approximation problem (**), if the conditions ii-iv hold then there exists a unique solution $u_h \in V_h$ solving (**).

**Proof:**
Variational Formulation

**Definition**

The **Ritz-Galerkin Approximation** is based on a finite-dimensional subspace $V_h \subset V$ and $F \in V'$. The problem is to find $u_h \in V_h$ so that

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(***)

**Theorem**

For the Ritz-Galerkin approximation problem (**), if the conditions ii-iv hold then there exists a unique solution $u_h \in V_h$ solving (**).

**Proof:**

This follows since $(V_h, a(\cdot, \cdot))$ is also a Hilbert space and we can again invoke the Riesz Representation Theorem to obtain representative $u_h$ that satisfies (**).
Variational Formulation

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**Theorem**

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This follows since $(V_h, a(\cdot, \cdot))$ is also a Hilbert space and we can again invoke the Riesz Representation Theorem to obtain representative $u_h$ that satisfies (**). ■
Lemma (Galerkin Orthogonality):

Let $u$ be solution of $(\ast)$ and $u_h$ the solution of $(\ast\ast)$, then the following orthogonality condition holds

$$a(u - u_h, v) = 0, \quad \forall v \in V_h.$$ 

Proof: Consider

$$a(u, v) = F[v], \quad v \in V$$

$$a(u_h, v) = F[v], \quad v \in V_h$$

Subtracting the equations we have

$$a(u - u_h, v) = 0, \quad v \in V_h.$$ 

$\blacksquare$
Lemma (Galerkin Orthogonality):
Let $u$ be solution of $(\ast)$ and $u_h$ the solution of $(\ast\ast)$, then the following orthogonality condition holds

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Lemma (Galerkin Orthogonality):

Let $u$ be solution of ($*$) and $u_h$ the solution of ($**$), then the following orthogonality condition holds

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Proof:
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Lemma (Galerkin Orthogonality):

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$$a(u, v) = F[v], \quad v \in V \quad a(u_h, v) = F[v], \quad v \in V_h$$
Lemma (Galerkin Orthogonality):
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Consider

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Lemma (Galerkin Orthogonality):
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Proof:
Consider

\[ a(u, v) = F[v], \quad v \in V \quad \quad \quad a(u_h, v) = F[v], \quad v \in V_h \]

Subtracting the equations we have

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Lemma (Galerkin Orthogonality):

Let $u$ be solution of $(\ast)$ and $u_h$ the solution of $(\ast\ast)$, then the following orthogonality condition holds

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Proof:

Consider

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$$a(u_h, v) = F[v], \quad v \in \mathcal{V}_h$$

Subtracting the equations we have

$$a(u - u_h, v) = 0, \quad v \in \mathcal{V}_h.$$
Lemma:

The solution of \((\ast\ast)\) for \(u_h \in V_h\) satisfies
\[
\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E.
\]

Lemma (Rayleigh-Ritz Method):
For the symmetric variational problem \((\ast\ast)\) the \(u_h\) minimizes the quadratic energy functional over all \(v \in V_h\) given by
\[
E[v] = a(v, v) - 2F[v].
\]
Lemma:
The solution of \((**)\) for \(u_h \in \mathcal{V}_h\) satisfies \(\|u - u_h\|_E = \min_{v \in \mathcal{V}_h} \|u - v\|_E\).
Lemma:
The solution of (**) for $u_h \in \mathcal{V}_h$ satisfies $\| u - u_h \|_E = \min_{v \in \mathcal{V}_h} \| u - v \|_E$.

Lemma (Rayleigh-Ritz Method):

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Lemma:

The solution of \((**)\) for \(u_h \in \mathcal{V}_h\) satisfies \(\|u - u_h\|_E = \min_{v \in \mathcal{V}_h} \|u - v\|_E\).

Lemma (Rayleigh-Ritz Method):

For the symmetric variational problem \((**)\) the \(u_h\) minimizes the quadratic energy functional over all \(v \in \mathcal{V}_h\) given by
Lemma:
The solution of (**) for \( u_h \in \mathcal{V}_h \) satisfies \( \| u - u_h \|_E = \min_{v \in \mathcal{V}_h} \| u - v \|_E \).

Lemma (Rayleigh-Ritz Method):
For the symmetric variational problem (**) the \( u_h \) minimizes the quadratic energy functional over all \( v \in \mathcal{V}_h \) given by
\[
E[v] = a(v, v) - 2F[v].
\]
Variational Formulation

Definition

A non-symmetric variational problem satisfies the following:

i. Given $F \in V'$, find $u$ satisfying

\[ a(u,v) = F[v], \quad \forall v \in V, \quad (***) \]

where

ii. $(H, (\cdot, \cdot))$ is a Hilbert space,

iii. $V$ is a subspace of $H$,

iv. $a(\cdot, \cdot)$ is a bilinear form (not-necessarily symmetric)

v. $a(\cdot, \cdot)$ is bounded on $H$ and coercive on $V$.
A non-symmetric variational problem satisfies the following

\[ a(u, v) = F(v), \quad \forall v \in V, \quad (\ast\ast\ast) \]

where

- \( H \) is a Hilbert space,
- \( V \) is a subspace of \( H \),
- \( a(\cdot, \cdot) \) is a bilinear form (not-necessarily symmetric)
- \( a \) is bounded on \( H \) and coercive on \( V \).
Definition

A **non-symmetric variational problem** satisfies the following

i. Given $F \in V'$, find $u$ satisfying

$$a(u, v) = F[v], \quad \forall v \in V, \quad (***)$$

where
Variational Formulation

Definition

A non-symmetric variational problem satisfies the following

i. Given $F \in \mathcal{V}'$, find $u$ satisfying

$$a(u, v) = F[v], \quad \forall v \in \mathcal{V}, \quad (***)$$

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Definition

A **non-symmetric variational problem** satisfies the following

i. Given $F \in \mathcal{V}'$, find $u$ satisfying

$$a(u, v) = F[v], \quad \forall v \in \mathcal{V}, \quad (***)$$

where

ii. $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space,

iii. $\mathcal{V}$ is a subspace of $\mathcal{H}$,
Variational Formulation

Definition

A **non-symmetric variational problem** satisfies the following

i Given \( F \in \mathcal{V}' \), find \( u \) satisfying

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A non-symmetric variational problem satisfies the following:

i. Given $F \in \mathcal{V}'$, find $u$ satisfying

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**Variational Formulation**

**Definition**

The Galerkin Approximation is based on a finite-dimensional subspace $V_h \subset V$ and $F \in V'$. The problem is to find $u_h \in V_h$ so that

$$a(u_h, v) = F[v], \quad \forall v \in V_h,$$

(∗∗∗)

We ideally would like to know the following:

1. Does a solution exist? Is the solution unique?
2. What error estimates hold for $u_h$ in approximating $u$?
3. What conditions might result in non-symmetric bilinear forms?

**Example:** Consider PDE

$$-u'' + u' + u = f, \quad x \in [0, 1],$$

$$u'(0) = u'(1) = 0.$$

A weak formulation is on $V = H^1([0, 1])$, $F[v] = (f, v)$, with

$$a(u, v) = \int_0^1 u' v' + u' v + uv \, dx,$$

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Finite Element Methods

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Variational Formulation

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Variational Formulation

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The Gallarkin Approximation is based on a finite-dimensional subspace \( \mathcal{V}_h \subset \mathcal{V} \) and \( F \in \mathcal{V}' \). The problem is to find \( u_h \in \mathcal{V}_h \) so that

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Finite Element Methods

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A contraction map is any mapping $T$ on a function space $V$ that satisfies for some $M < 1$:
\[
\|Tv_1 - Tv_2\| \leq M\|v_1 - v_2\|.
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A fixed point $u$ of $T$ is any function satisfying $u = Tu$.

Lemma (Fix Point Theorem) If $T$ is a contraction map on a Banach space $V$ then there exists a unique fixed point $u$ satisfying $u = Tu$. 

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Definition (Contraction Mapping)

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Variational Formulation

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We show uniqueness first, then existence. Suppose $Tv_1 = v_1$ and $Tv_2 = v_2$, then by the contraction principle

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where $0 \leq M < 1$.

By the fix-point property

$$\|v_1 - v_2\| \leq M\|v_1 - v_2\|.$$ 

This implies $\|v_1 - v_2\| = 0 \rightarrow v_1 = v_2$.

We next show existence.
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For a given $v_0 \in V$ define the generated sequence as $v_{k+1} = Tv_k$. This satisfies

$$\|v_k - v_{k-1}\| \leq M^{k-1} \|v_1 - v_0\|.$$ 

For any $N > n$ we have

$$\|v_N - v_n\| \leq \sum_{k=n+1}^{N} M^{k-1} \|v_1 - v_0\| \leq M^{n-1} \|Tv_0 - v_0\|.$$ 

This shows $\{v_k\}$ forms a Cauchy sequence and by completeness we have there exists $v^* \in V$ so that $v^* = \lim_{n \to \infty} v_n = \lim_{n \to \infty} Tv_n = T(\lim_{n \to \infty} v_n) = Tv^*$. This establishes existence of a fixed point for $T$. ■
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Variational Formulation

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Theorem (Lax-Milgram)

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**Significance:** This establishes for variational problems the *existence* and *uniqueness* of the solution $u$. 

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Finite Element Methods

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Variational Formulation

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Given a Hilbert space \((\mathcal{V}, (\cdot, \cdot))\), a continuous, coercive bilinear form \(a(\cdot, \cdot)\) (not necessarily symmetric), and \(F \in \mathcal{V}'\), there exists a unique \(u \in \mathcal{V}\) so that

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**Implications:** Also shows for the Galerkin approximations for the finite-dimensional problems the existence and uniqueness of solution \(u_h\).
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Define the operator \(Au\) which has action on a function \(v \in V\) as

\[ Au[v] = a(u, v). \]

Properties of \(a\) imply \(Au\) is linear, bounded, and has norm

\[ \|Au\|_{V'} \leq C \|u\|_V < \infty, \]

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\]

\[
= \|v_1 - \rho(\tau Av_1)\|^2,
\]

\[
\leq \|v_1\|^2 - 2\rho a(v_1, v) + \rho^2 \|\tau Av_1\|^2,
\]

\[
\leq \|v_1\|^2 - 2\rho a(v_1, v) + \rho^2 C^2 \|v_1\|^2,
\]

\[
(1 - 2\rho a + \rho^2 C^2) \|v_1\|^2,
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(A bounded, \(\tau\) isometric)

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\]

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\[
1 - 2\rho \alpha + \rho^2 C^2 < 1 \rightarrow \rho (\rho C^2 - 2\alpha) < 0. \quad (1)
\]
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1 - 2\rho\alpha + \rho^2 C^2 < 1 \rightarrow \rho (\rho C^2 - 2\alpha) < 0.
\]  

This is satisfied for \(\rho \in (0, 2\alpha/C^2)\) giving \(M < 1\).
Theorem (Lax-Milgram)

Given a Hilbert space \((V, (\cdot, \cdot))\), a continuous, coercive bilinear form \(a(\cdot, \cdot)\) (not necessarily symmetric), and \(F \in V'\), there exists a unique \(u \in V\) so that

\[
a(u, v) = F[v], \quad \forall v \in V.
\]

Proof (continued):

\[
\|Tv_1 - Tv_2\|^2 \leq (1 - 2\rho \alpha + \rho^2 C^2) \|v\|^2 \quad \text{(A bounded, } \tau \text{ isometric)}
\]
\[
= (1 - 2\rho \alpha + \rho^2 C^2) \|v_1 - v_2\|^2
\]
\[
= M^2 \|v_1 - v_2\|^2.
\]

We need

\[
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This is satisfied for \(\rho \in (0, 2\alpha/C^2)\) giving \(M < 1\). By the contraction principle we obtain the results. ■
Theorem (Céa)
Suppose we have the conditions hold for the variational problems (⋆) or (⋆⋆).
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Suppose we have the conditions hold for the variational problems (\(*\)) or (\(*\,*\,*\)). For the bilinear form $a(\cdot, \cdot)$, let $C$ denote the continuity constant in the boundedness condition and $\alpha$ denote the coercivity parameter.

The following error bound holds for the Galerkin approximation

$$
\|u - u_h\|_{V} \leq C \alpha \min_{v \in V_h} \|u - v\|_{V}.
$$

Significance:

This shows the solution $u_h$ obtained from the Galerkin approximation is bounded by all approximations in the space $V$ when measuring errors in the Hilbert-space norm. This will become the basis for further estimates on the accuracy of Finite Element Methods.

Proof:

By subtracting the variational problems for the exact and Galerkin approximation we obtain

$$
a(u - u_h, v) = 0 \quad \forall v \in V_h.
$$

For all $v \in V_h$ we have

$$
\alpha \|u - u_h\|^2_V \leq a(u - u_h, u - u_h) \quad \text{(by coercivity)}.
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Variational Formulation

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Suppose we have the conditions hold for the variational problems (\(\ast\)) or (\(\ast\ast\ast\)). For the bilinear form \(a(\cdot, \cdot)\), let \(C\) denote the continuity constant in the boundedness condition and \(\alpha\) denote the coercivity parameter. The following error bound holds for the Galerkin approximation

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Paul J. Atzberger, UCSB
Finite Element Methods
http://atzberger.org/
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By subtracting the variational problems for the exact and Galerkin approximation we obtain

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Variational Formulation

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\[
\| u - u_h \|_V \leq \frac{C}{\alpha} \min_{v \in \mathcal{V}_h} \| u - v \|_V.
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Significance: This shows the solution \( u_h \) obtain from the Galerkin approximation is bounded by all approximations in the space \( \mathcal{V} \) when measuring errors in the Hilbert-space norm. This will become the basis for further estimates on the accuracy of Finite Element Methods.

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By subtracting the variational problems for the exact and Galerkin approximation we obtain

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For all \( v \in \mathcal{V}_h \) we have

\[
\alpha \| u - u_h \|_V^2 \leq a(u - u_h, u - u_h) \quad \text{(by coercivity)}
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Suppose we have the conditions hold for the variational problems (\(\ast\)) or (\(\ast\ast\ast\)). For the bilinear form \(a(\cdot, \cdot)\), let \(C\) denote the continuity constant in the boundedness condition and \(\alpha\) denote the coercitivity parameter. The following error bound holds for the Galerkin approximation

\[
\|u - u_h\|_V \leq C \min_{v \in V_h} \|u - v\|_V.
\]

Proof (continued):
Variational Formulation

Theorem (Céa)

Suppose we have the conditions hold for the variational problems (∗) or (∗∗∗). For the bilinear form $a(\cdot, \cdot)$, let $C$ denote the continuity constant in the boundedness condition and $\alpha$ denote the coercitivity parameter. The following error bound holds for the Galerkin approximation

$$
\|u - u_h\|_V \leq C \alpha \min_{\nu \in V_h} \|u - \nu\|_V.
$$

Proof (continued):

$$
\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) \quad \text{(by coercivity)}
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\[
\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V.
\]

**Proof (continued):**

\[
\begin{align*}
\alpha \|u - u_h\|^2_V & \leq a(u - u_h, u - u_h) \quad \text{(by coercivity)} \\
& = a(u - u_h, u - v) + a(u - u_h, v - u_h)
\end{align*}
\]
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Suppose we have the conditions hold for the variational problems \((*)\) or \((***)\). For the bilinear form \(a(\cdot, \cdot)\), let \(C\) denote the continuity constant in the boundedness condition and \(\alpha\) denote the coercitivity parameter. The following error bound holds for the Galerkin approximation

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= a(u - u_h, u - v) + a(u - u_h, v - u_h)
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\[
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\]

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\[
\alpha \| u - u_h \|_{\mathcal{V}}^2 \leq a(u - u_h, u - u_h) \quad \text{(by coercivity)}
\]
\[
= a(u - u_h, u - v) + a(u - u_h, v - u_h)
\]
\[
= a(u - u_h, u - v), \quad \text{(since } v - u_h \in \mathcal{V}_h \text{)}
\]
\[
\leq C \| u - u_h \|_{\mathcal{V}} \| u - v \|_{\mathcal{V}} \quad \text{(by continuity)} .
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By dividing through we obtain for all \( v \in \mathcal{V}_h \)
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\begin{align*}
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& = a(u - u_h, u - v) + a(u - u_h, v - u_h) \\
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\[
\| u - u_h \|_V \leq \frac{C}{\alpha} \| u - v \|_V.
\]

This implies (since \( V_h \) is closed)

\[
\| u - u_h \|_V \leq \frac{C}{\alpha} \inf_{v \in V_h} \| u - v \|_V = \frac{C}{\alpha} \min_{v \in V_h} \| u - v \|_V.
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Suppose we have the conditions hold for the variational problems (∗) or (∗∗∗). For the bilinear form $a(\cdot, \cdot)$, let $C$ denote the continuity constant in the boundedness condition and $\alpha$ denote the coercitivity parameter. The following error bound holds for the Galerkin approximation

$$\|u - u_h\|_V \leq C \frac{\alpha}{\min_{v \in V_h} \|u - v\|_V}.$$

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\[\blacksquare\]