Finite Element Spaces

Paul J. Atzberger

206D: Finite Element Methods
University of California Santa Barbara
Definition

Consider

An element domain \( K \subseteq \mathbb{R}^n \) that is a bounded closed set with non-empty interior and piece-wise smooth boundary.

ii The shape functions \( P \) consist of a finite-dimensional space of functions on \( K \).

iii The nodal variables \( N = \{N_1, N_2, \ldots, N_k\} \) are any basis of the dual space \( P' \).

A finite element is the triple \((K, P, N)\).

This definition of FEM is due to Ciarlet. Sometimes also denoted by \((T, \Pi, \Sigma)\).

Definition

For a finite element \((K, P, N)\), the nodal basis \( \{\phi_i\}_{i=1}^k \) of \( P \) is the collection of functions for which \( N_i(\phi_j) = \delta_{ij} \).

Example: Consider the finite element with \( K = [0, 1] \) and \( P \) with linear polynomial basis with \( N = \{N_1, N_2\} \), where \( N_1(v) = v(0) \) and \( N_2(v) = v(1) \).

Then \( \phi_1(x) = 1 - x \) and \( \phi_2(x) = x \).
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**Definition**

The nodal variables $\mathcal{N}$ are said to **determine** members of $\mathcal{P}$ if for $\psi \in \mathcal{P}$ we have $N(\psi) = 0 \ \forall N \in \mathcal{N}$ implies $\psi \equiv 0$. 

**Lemma**

The following statements are equivalent

i. For $v \in \mathcal{P}$ with $N_i(v) = 0$, $\forall i$, then $v \equiv 0$.

ii. The collection $\{N_1, N_2, \ldots, N_k\}$ is a basis for $\mathcal{P}'$.

**Proof:**

Suppose $\{\phi_i\}$ are a basis for $\mathcal{P}$. The $\{N_i\}$ are basis for $\mathcal{P}'$ iff for any $L \in \mathcal{P}'$ we have $L = \alpha_1 N_1 + \ldots + \alpha_d N_d$ and $L \equiv 0$ implies $\alpha_i = 0$.

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**Proof:** Suppose \( \{ \phi_i \} \) are a basis for \( \mathcal{P} \). The \( \{ N_i \} \) are basis for \( \mathcal{P}' \) iff for any \( L \in \mathcal{P}' \) we have

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Let matrix $B_{ij} = N_j(\phi_i)$, then above corresponds to solving $B\alpha = y$, where $y_i = L(\phi_i)$,
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Let matrix $B_{ij} = N_j(\phi_i)$, then above corresponds to solving $B\alpha = y$, where $y_i = L(\phi_i)$, so

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Consider $v \in P$ with $v = \sum_j \beta_j \phi_j$. 
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\( \text{(ii)} \iff B \) is invertible.

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Consider \( v \in \mathcal{P} \) with \( v = \sum_j \beta_j \phi_j \). If \( N_i(v) = 0 \), then \( \sum_j \beta_j N_j(\phi_i) = 0 \). The \( v \equiv 0 \) if \( \beta_j = 0 \).

Let matrix \( C_{ij} = N_i(\phi_j) \), \( x_j = \beta_j \), then above corresponds to \( C x = 0 \Rightarrow x = 0 \),
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Let matrix $C_{ij} = N_i(\phi_j)$, $x_j = \beta_j$, then above corresponds to $Cx = 0 \Rightarrow x = 0$, so

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The \( C = B^T \) so it follows (i) \( \iff \) (ii).
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Finite Elements

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Definition

We call **admissible** a partition of \( \Omega \) into \( T = \{ T_1, T_2, \ldots, T_M \} \) into triangular or quadrilateral elements if

1. The \( T_i \) form a partition \( \Omega = \bigcup_{i=1}^{M} T_i \).
2. For \( i \neq j \) the \( T_i \cap T_j \) only intersect along an edge or vertex.
Conforming Finite Elements are those that generate a space $S$ with $S \subset \mathcal{V}$. The generated space $S$ is a subspace of the $\mathcal{V}$ used for the weak formulation.

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[Diagram of admissible and inadmissible triangulations]
Finite Elements

Theorem

For a bounded domain \( \Omega \), admissible partition, and \( k \geq 1 \), a piecewise infinitely differentiable function \( v : \overline{\Omega} \rightarrow \mathbb{R} \) belongs to \( H^k(\Omega) \) if and only if \( v \in C^{k-1}(\overline{\Omega}) \).
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For a bounded domain Ω, admissible partition, and \( k \geq 1 \), a piecewise infinitely differentiable function \( v : \overline{\Omega} \rightarrow \mathbb{R} \) belongs to \( H^k(\Omega) \) if and only if \( v \in C^{k-1}(\overline{\Omega}) \).

Proof: We show this for the case \( k = 1 \), and for simplicity \( \mathbb{R}^2 \).
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Paul J. Atzberger, UCSB

Finite Element Methods

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$$(w_i, \phi)_0 = \int_\Omega w_i \phi dx = \sum_j \int_{T_j} \phi \partial_i v dx = \sum_j \left( \int_{\partial T_j} \phi v dx - \int_{T_j} v \partial_i \phi dx \right) = -(v, \partial_i \phi)_0.$$
Finite Elements

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The boundary term vanishes since $\phi(x) = 0$ for $x \in \partial \Omega$ and internal boundary terms cancel.
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Finite Elements

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$(\Rightarrow)$ Let $v \in H^1(\Omega)$. Consider a neighborhood of an edge and use coordinates based on rotation so the edge lies along the $y$-axis as interval $[y_1 - \delta, y_2 + \delta]$, $\delta > 0$. 

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This means the function $\Psi(x)$ is continuous, in particular at $x = 0$. Since $y_1, y_2$ can be chosen arbitrary with $y_1 < y_2$, $\Psi$ can only be continuous if $v$ is continuous at the edge, $\Rightarrow v \in C^0$.

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Significance: This shows that provided our elements are smooth piecewise and have derivatives $C^{k-1}$ across edges, we obtain conforming elements for $V = H^k(\Omega)$.

Example: While hat-functions are only $C^0$, they provide elements conforming to $V = H^1(\Omega)$. Allows for approximating in weak form second-order PDEs.

Example: Elements with $C^1$-regularity across edges are sufficient to conform to $V = H^2(\Omega)$. Allows for approximating in weak form fourth-order PDEs.
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Definitions

Two elements are congruent if they can be rigidly transformed into each other (allowing reflections).

A partition by elements is called regular if all the elements are congruent (same type and shape).

The space of polynomials of degree $t$ with $x \in \mathbb{R}^n$ is denoted by $P_t = \{ u | u(x) = \sum |\alpha| \leq t c_\alpha x^\alpha \}$.

Elements with complete polynomials refers to shape spaces using all polynomials with degree $\leq t$.

Conforming finite elements are those that generate function spaces contained in the Sobolev space of the weak formulation.

Other shape spaces, partition types, and non-conforming finite elements are also possible.
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Practical Methods: A Few Considerations

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Other shape spaces, partition types, and non-conforming finite elements are also possible.
Consider partition of the domain into triangular elements $\mathcal{T}$.
Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $T$.

Lemma

Consider triangle $T$ with $z_1, z_2, \ldots, z_s, s = 1 + 2 + \cdots + (t+1)$ nodes lying on the lines depicted. For every $f \in C(T)$ there is a unique polynomial $p$ of degree $\leq t$ satisfying interpolation $p(z_i) = f(z_i), 1 \leq i \leq s$.

Proof: We proceed by induction. Clearly, in the case of $t = 0$ when $s = 1$ we have interpolation by the constant polynomials. Now if the interpolation for $t-1$ holds, we prove it holds for $t$.

Let $p_1$ be the univariate Lagrange polynomial interpolating the $t+1$ points on the x-axis. Consider the sub-triangle neglecting the points on the x-axis. Let $p_2$ be the interpolating polynomial for these points with $p_2(z_i) = (f(z_i) - p_1(z_i))/y_i, 1 \leq i \leq s-(t+1)$.

The polynomial $q(x, y) = p_1(x) + yp_2(x, y)$ interpolates all points. Uniqueness as exercise (use holds for degree $t-1$). ■
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**Lemma**

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For every $f \in C^0(T)$ there is a unique polynomial $p$ of degree $\leq t$ satisfying interpolation $p(z_i) = f(z_i)$, $1 \leq i \leq s$.

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- Linear triangular element $M_0^1$
  - $u \in C^0(\Omega)$
  - $\Pi_{\text{ref}} = \mathcal{P}_1$, $\dim \Pi_{\text{ref}} = 3$

- Quadratic triangular element $M_0^2$
  - $u \in C^0(\Omega)$
  - $\Pi_{\text{ref}} = \mathcal{P}_2$, $\dim \Pi_{\text{ref}} = 6$

- Cubic triangular element $M_0^3$
  - $u \in C^0(\Omega)$
  - $\Pi_{\text{ref}} = \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 10$

- Function value prescribed
- Function value and 1st derivative prescribed
- Function value and 1st and 2nd derivatives prescribed
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D. Braess 2007
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Consider partition of the domain into triangular elements $T$.

**Lemma**

Consider triangle $T$ with $z_1, z_2, \ldots, z_s$, $s = 1 + 2 + \cdots (t + 1)$ nodes lying on the lines depicted. For every $f \in C(T)$ there is a unique polynomial $p$ of degree $\leq t$ satisfying interpolation

$$p(z_i) = f(z_i), \quad 1 \leq i \leq s.$$  

**Proof:** We proceed by induction. Clearly, in the case of $t = 0$ when $s = 1$ we have interpolation by the constant polynomials. Now if the interpolation for $t - 1$ holds, we prove it holds for $t$. 

Linear triangular element $M_0^1$

$$u \in C^0(\Omega)$$

$$\Pi_{\text{ref}} = P_1, \quad \dim \Pi_{\text{ref}} = 3$$

Quadratic triangular element $M_0^2$

$$u \in C^0(\Omega)$$

$$\Pi_{\text{ref}} = P_2, \quad \dim \Pi_{\text{ref}} = 6$$

Cubic triangular element $M_0^3$

$$u \in C^0(\Omega)$$

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Linear triangular element $\mathcal{M}_0^1$

\begin{align*}
    u &\in C^0(\Omega) \\
    \Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3
\end{align*}

Quadratic triangular element $\mathcal{M}_0^2$

\begin{align*}
    u &\in C^0(\Omega) \\
    \Pi_{\text{ref}} = \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 6
\end{align*}

Cubic triangular element $\mathcal{M}_0^3$

\begin{align*}
    u &\in C^0(\Omega) \\
    \Pi_{\text{ref}} = \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 10
\end{align*}

- Function value prescribed
- Function value and 1st derivative prescribed
- Function value and 1st and 2nd derivatives prescribed
- Normal derivative prescribed

D. Braess 2007
Consider partition of the domain into triangular elements $T$.

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Consider triangle $T$ with $z_1, z_2, \ldots, z_s$, $s = 1 + 2 + \cdots (t + 1)$ nodes lying on the lines depicted. For every $f \in C(T)$ there is a unique polynomial $p$ of degree $\leq t$ satisfying interpolation

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- Quadratic triangular element $M_0^2$
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  - $\Pi_{ref} = \mathcal{P}_2$, $\dim \Pi_{ref} = 6$

- Cubic triangular element $M_0^3$
  - $u \in C^0(\Omega)$
  - $\Pi_{ref} = \mathcal{P}_3$, $\dim \Pi_{ref} = 10$

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Triangular Finite Elements: Lagrange Elements

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Linear triangular element $M_0^1$
\[ u \in C^0(\Omega) \]
\[ \Pi_{\text{ref}} = \mathcal{P}_1, \ \dim \Pi_{\text{ref}} = 3 \]

Quadratic triangular element $M_0^2$
\[ u \in C^0(\Omega) \]
\[ \Pi_{\text{ref}} = \mathcal{P}_2, \ \dim \Pi_{\text{ref}} = 6 \]

Cubic triangular element $M_0^3$
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Consider partition of the domain into triangular elements $\mathcal{T}$.

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The polynomial $q(x, y) = p_1(x) + yp_2(x, y)$ interpolates all points.
Consider partition of the domain into triangular elements $T$.

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Consider partition of the domain into triangular elements $\mathcal{T}$.

\[ M_k^0 := \left\{ v \in L^2(\Omega); \quad v|_T \in P_t \text{ for every } T \in \mathcal{T} \right\} \]

\[ M_0^0 := M_k^0 \cap C^0(\Omega) = M_k \cap H^1(\Omega) \]

\[ M_0^1, 0 := M_k \cap H^1_0(\Omega). \]

The $M_k^0$ provide $C^0$ elements $\subset H^1$.

Note: Shared common nodes at vertices ensures the continuity.

$M_k^0$ is called the conforming $P_k$ element.

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\[ \Pi_{\text{ref}} = P_2, \quad \dim \Pi_{\text{ref}} = 6 \]

Cubic triangular element $M_0^3$

\[ u \in C^0(\Omega) \]

\[ \Pi_{\text{ref}} = P_3, \quad \dim \Pi_{\text{ref}} = 10 \]

- Function value prescribed
- Function value and 1st derivative prescribed
- Function value and 1st and 2nd derivatives prescribed
- Normal derivative prescribed

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Consider partition of the domain into triangular elements $\mathcal{T}$.

- Linear triangular element $\mathcal{M}_0^1$
  - $u \in C^0(\Omega)$
  - $\Pi_{ref} = \mathcal{P}_1$, $\dim \Pi_{ref} = 3$

- Quadratic triangular element $\mathcal{M}_0^2$
  - $u \in C^0(\Omega)$
  - $\Pi_{ref} = \mathcal{P}_2$, $\dim \Pi_{ref} = 6$

- Cubic triangular element $\mathcal{M}_0^3$
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Consider partition of the domain into triangular elements $\mathcal{T}$.

**Definition**

\[ \mathcal{M}^k := \mathcal{M}_k(\mathcal{T}) := \{ v \in L^2(\Omega); v|_T \in \mathcal{P}_t \text{ for every } T \in \mathcal{T} \} \]

- Linear triangular element $\mathcal{M}_0^1$
  \[ u \in C^0(\Omega) \]
  \[ \Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3 \]

- Quadratic triangular element $\mathcal{M}_0^2$
  \[ u \in C^0(\Omega) \]
  \[ \Pi_{\text{ref}} = \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 6 \]

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Consider partition of the domain into triangular elements \( T \).

**Definition**

\[
M^k := M_k(T) := \{ v \in L^2(\Omega); \, v|_T \in P_t \text{ for every } T \in T \},
\]

\[
M^k_0 := M_k(T) \cap C^0(\Omega) = M^k \cap H^1(\Omega),
\]

\[
M^k_{0,0} := M^k \cap H^1_0(\Omega).
\]

- Linear triangular element \( M^1_0 \)
  - \( u \in C^0(\Omega) \)
  - \( \Pi_{\text{ref}} = P_1, \quad \text{dim} \Pi_{\text{ref}} = 3 \)

- Quadratic triangular element \( M^2_0 \)
  - \( u \in C^0(\Omega) \)
  - \( \Pi_{\text{ref}} = P_2, \quad \text{dim} \Pi_{\text{ref}} = 6 \)

- Cubic triangular element \( M^3_0 \)
  - \( u \in C^0(\Omega) \)
  - \( \Pi_{\text{ref}} = P_3, \quad \text{dim} \Pi_{\text{ref}} = 10 \)

- Function value prescribed
- Function value and 1st derivative prescribed
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Finite Element Methods

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Triangular Finite Elements: Lagrange Elements

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$\mathcal{M}^k_0 := \mathcal{M}_k(\mathcal{T}) \cap C^0(\Omega) = \mathcal{M}^k \cap H^1(\Omega)$

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- Linear triangular element $\mathcal{M}^1_0$
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  $u \in C^0(\Omega)$
  $\Pi_{\text{ref}} = \mathcal{P}_2$, \ \dim \Pi_{\text{ref}} = 6$

- Cubic triangular element $\mathcal{M}^3_0$
  $u \in C^0(\Omega)$
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**Definition**

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\mathcal{M}^k := \mathcal{M}_k(\mathcal{T}) := \{ v \in L^2(\Omega); v|_T \in P_t \text{ for every } T \in \mathcal{T} \} \\
\mathcal{M}_0^k := \mathcal{M}_k(\mathcal{T}) \cap C^0(\Omega) = \mathcal{M}^k \cap H^1(\Omega) \\
\mathcal{M}_{0,0}^k := \mathcal{M}^k \cap H^1_0(\Omega).
\]

The $\mathcal{M}_0^k$ provide $C^0$ elements $\subset H^1$.

**Note:** Shared common nodes at vertices ensures the continuity.
Consider partition of the domain into triangular elements $\mathcal{T}$.

**Definition**

\[
\mathcal{M}^k := \mathcal{M}_k(\mathcal{T}) := \{ v \in L^2(\Omega); \ v|_{\mathcal{T}} \in \mathcal{P}_t \text{ for every } \mathcal{T} \in \mathcal{T} \}
\]

\[
\mathcal{M}^k_0 := \mathcal{M}_k(\mathcal{T}) \cap C^0(\Omega) = \mathcal{M}^k \cap H^1(\Omega)
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$\mathcal{M}^k_0$ is called the **conforming $P_k$ element**.
Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $T$.

**Definition**

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\mathcal{M}^k := \mathcal{M}_k(T) := \{ v \in L^2(\Omega); v|_T \in \mathcal{P}_t \text{ for every } T \in T \}
\]

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\mathcal{M}^k_0 := \mathcal{M}_k(T) \cap C^0(\Omega) = \mathcal{M}^k \cap H^1(\Omega)
\]

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The $\mathcal{M}^k_0$ provide $C^0$ elements $\subset H^1$.

**Note:** Shared common nodes at vertices ensures the continuity.

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Consider partition of the domain into triangular elements $\mathcal{T}$.

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Nodal variables are $N_j(u) = u(z_j)$, so also called **Lagrange elements**.
Triangular Finite Elements: $C^1$ Regularity

More challenging to obtain elements with $C^1$ regularity.

Argyris element:
Uses $P_5$ which has dim $P_5 = 21$.
Values given of all derivatives up to order 2 at the vertices.
However, this is only $3 \times 6 = 18$ DOF.
Determine 3 DOF from normal derivative at edge centers.

Bell element:
Uses $\tilde{P}_5 = P_5 \mathbb{Q}$ which has dim $\tilde{P}_5 = 18$.
$\tilde{P}_5$ restricted to polynomials having normal derivatives along the edges only of degree 4, ($\partial_n p(x_e) \in P_4$).
Values given of all derivatives up to order 2 at the vertices.

Hsieh-Clough-Tocher element:
Macroelement approach.
Subdivide the triangle into three subtriangles.
Use $S$ piecewise cubic polynomials on each subtriangle, dim $S = 12$.
Values given of function and first derivative at vertices.
Values of the normal derivative at edge centers.
Bernstein-Bézier representation of polynomials used to handle derivatives along element boundaries.
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- **Argyris triangle**
  - $u \in C^1(\Omega)$
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- **Bell triangle**
  - $u \in C^1(\Omega)$
  - $\Pi_{\text{ref}} \subset P_5$, $\partial_n u|_{\partial T_i} \in P_3$, $\dim \Pi_{\text{ref}} = 18$

- **Hsieh–Clough–Tocher element**
  - $u \in C^1(\Omega)$
  - $T = \bigcup_{i=1}^3 K_i$, $u|_{K_i} \in P_3$, $\dim \Pi_{\text{ref}} = 12$

- Function value prescribed
- Function value and 1st derivative prescribed
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Argyris triangle

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Bell triangle

$u \in C^1(\Omega)$

$\Pi_{\text{ref}} \subset \mathcal{P}_5, \quad \partial_n u|_{\partial T} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 18$

Hsieh–Clough–Tocher element

$u \in C^1(\Omega)$

$T = \bigcup_{i=1}^{3} K_i, \quad u|_{K_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 12$

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- **Bell triangle**
  - $u \in C^1(\Omega)$
  - $\Pi_{\text{ref}} \subset \mathcal{P}_5$, $\partial_u |_{\partial \Omega} \in \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 18$

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  - $u \in C^1(\Omega)$
  - $T = \bigcup_{i=1}^3 K_i$, $u|_{K_i} \in \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 12$

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More challenging to obtain elements with $C^1$ regularity.

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Uses $\tilde{\mathcal{P}}_5 = \mathcal{P}_5 \mathcal{Q}$ which has dim $\tilde{\mathcal{P}}_5 = 18$.
Values given of all derivatives up to order 2 at the vertices.

**Hsieh–Clough–Tocher element:**
Macroelement approach.
Subdivide the triangle into three subtriangles.
Use $S$ piecewise cubic polynomials on each subtriangle, dim $S = 12$.
Values given of function and first derivative at vertices.
Values of the normal derivative at edge centers.

Bernstein–Bézier representation of polynomials used to handle derivatives along element boundaries.

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Triangular Finite Elements: $C^1$ Regularity

More challenging to obtain elements with $C^1$ regularity.

**Argyris element:**
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Determine 3 DOF from normal derivative at edge centers.

$\text{Argyris triangle}$
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$\text{Bell triangle}$
\[ u \in C^1(\Omega) \]
\[ \Pi_{\text{ref}} \subset \mathcal{P}_5, \quad \partial_n u|_{\partial \Omega} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 18 \]

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\[ u \in C^1(\Omega) \]
\[ T = \bigcup_{i=1}^3 K_i, \quad u|_{K_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 12 \]

- Function value prescribed
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Bernstein-Bézier representation of polynomials used to handle derivatives along element boundaries.
A tensor-product basis generated by \( \{ \varphi_k \} \) for \( x \in \mathbb{R}^n \) is given by:

\[
\tilde{P}_k := \{ u(x) | u(x_1, x_2, \ldots, x_n) = \sum_{1 \leq j_1, \ldots, j_n \leq t} c_{j_1} \varphi_{j_1}(x_1) \cdot \varphi_{j_2}(x_2) \cdots \varphi_{j_n}(x_n) \}.
\]

The polynomial tensor-product basis of degree \( t \) is:

\[
Q_t := \{ u | u(x) = \sum_{\max \alpha \leq t} c_{\alpha} x_\alpha \}.
\]

The space \( Q_1 \) gives bilinear interpolation of nodal values. In fact, \( Q_1 = \{ u \in C_0(\Omega) | v | T \in P_2, \text{along edges} v | \partial T \in P_1 \} \).
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\( \Pi_{\text{ref}} \subset P_2, \ u|_{\partial T_i} \in P_1, \ \dim \Pi_{\text{ref}} = 4 \)

Serendipity element
\( u \in C^0(\Omega) \)
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- Function value prescribed
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Quadrilateral Finite Elements

Serendipity Element:
\[ S_{sd} = \{ u \in P_3 | u |_{\partial T} \in P_2 \}, \] which has \( \dim S_{sd} = 8 \).

\[
\begin{align*}
p(x, y) &= c_0 + c_1 x + c_2 y + c_3 xy + c_4 (x^2 - 1)(y - 1) + c_5 (x^2 - 1)(y + 1) + c_6 (x - 1)(y^2 - 1) + c_7 (x + 1)(y^2 - 1) \\
\end{align*}
\]

Nodal locations are vertices of rectangle and edge mid-points.

9-Point Element:
Consider \( S_9 = S_{sd} \oplus \{ c_8 (x^2 - 1)(y^2 - 1) \} \). Nodal locations are vertices of rectangle and edge mid-points.

Add nodal location at the center of the rectangle.

6-Point Element:
Consider \( S_{sd} \mathcal{Q} \) for some \( \mathcal{Q} \) of polynomials terms.

For \( \mathcal{Q} = \{ c_4 (x^2 - 1)(y - 1) \oplus c_5 (x^2 - 1)(y + 1) \} \), drop midpoint nodes on edges with \( y = \pm 1 \).

For \( \mathcal{Q} = \{ c_6 (x - 1)(y^2 - 1) \oplus c_7 (x + 1)(y^2 - 1) \} \), drop midpoint nodes on edges with \( x = \pm 1 \).

Bilinear quadrilateral element \( Q_1 \)
\[ u \in C^0(\Omega) \]
\[ \Pi_{ref} \subset P_2, \ u|_{\partial T} \in P_1, \ \dim \Pi_{ref} = 4 \]

Serendipity element
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Bilinear quadrilateral element $Q_1$

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$\Pi_{\text{ref}} \subset P_2$, $u_{\text{\partial } T_i} \in P_1$, $\dim \Pi_{\text{ref}} = 4$

Serendipity element

$u \in C^0(\Omega)$

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Serendipity element

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$\Pi_{\text{ref}} \subset \mathcal{P}_3$, $u|_{\partial T_i} \in \mathcal{P}_2$, $\dim \Pi_{\text{ref}} = 8$

- Function value prescribed
- Function value and 1st derivative prescribed
- Function value and 1st and 2nd derivatives prescribed
- Normal derivative prescribed

D. Braess 2007
Quadrilateral Finite Elements

Serendipity Element:
Consider \( S_{sd} = \{ u \in P_3 \mid u|_{\partial T} \in P_2 \} \), which has \( \dim S_{sd} = 8 \).
\[
\begin{align*}
p(x, y) &= c_0 + c_1 x + c_2 y + c_3 xy \\
&\quad + c_4 (x^2 - 1)(y - 1) + c_5 (x^2 - 1)(y + 1) \\
&\quad + c_6 (x - 1)(y^2 - 1) + c_7 (x + 1)(y^2 - 1).
\end{align*}
\]
Nodal locations are vertices of rectangle and edge mid-points.

9-Point Element:

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Consider \( S_{9} = S_{sd} \bigoplus \{ c_8 (x^2 - 1)(y^2 - 1) \} \).
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Bilinear quadrilateral element \( Q_1 \)
\[
\begin{align*}
&u \in C^0(\Omega) \\
&\Pi_{ref} \subset P_2, \ u|_{\partial T_i} \in P_1, \ \dim \Pi_{ref} = 4
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\]

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D. Braess 2007
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D. Braess 2007
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D. Braess 2007
Quadrilateral Finite Elements

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**6-Point Element:**
Quadrilateral Finite Elements

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Consider $S_{sd} \setminus Q$ for some $Q$ of polynomials terms.

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D. Braess 2007
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Consider $S_{sd} \setminus Q$ for some $Q$ of polynomials terms. For $Q = \{ c_4 (x^2 - 1)(y - 1) \oplus c_5 (x^2 - 1)(y + 1) \}$, drop midpoint nodes on edges with $y = \pm 1$. 
**Quadrilateral Finite Elements**

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Affine Families of Elements

Definition

We define for canonical representation a reference element \((T_{\text{ref}}, \Pi_{\text{ref}}, \Sigma_{\text{ref}})\). A collection of finite element spaces \(S_h\) for partitions \(T_h \subset \Omega \subset \mathbb{R}^d\) is called an affine family if for every \(T_j \in T_h\) there exists an affine map \(F_j : T_{\text{ref}} \rightarrow T_j\) so that when \(v \in S_h\) when restricted to \(T_j\) is of the form \(v(x) = p(F_j^{-1}x)\) with \(p \in \Pi_{\text{ref}}\).

The finite elements \(M_{k0}\) are an affine family. The quadrilateral elements we defined using nodal values give affine families. However, the Argyris elements are not since they involve normal derivatives.
Affine Families of Elements

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We define for canonical representation a **reference element** \((\mathcal{T}_{\text{ref}}, \Pi_{\text{ref}}, \Sigma_{\text{ref}})\).

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\[
v(x) = p(F_j^{-1}(x)) \quad \text{with} \quad p \in \Pi_{ref}.
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Application to Elliptic PDEs.
Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

\[ \nabla^2 u = -g, \quad x \in \Omega \\
\left. u \right|_{\partial \Omega} = f, \quad x \in \partial \Omega . \]

\[ \langle a(u, v) = -\langle g, v \rangle, \quad v \in S \rangle \]

(RG-Approximation)

Motivations:
- Steady-state heat equation
- Electrostatics
- Incompressibility constraints

Finite Element Approximation Steps:
- Select element type for generating a space \( S \).
- Mesh the domain to obtain a collection of elements.
- Calculate the stiffness matrix and load vector using weak form.
- Solve the linear system \( K u = f \).

Paul J. Atzberger, UCSB
http://atzberger.org/
Poisson Equation as Model Problem:

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

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Steady-state heat equation, electrostatics, incompressibility constraints.

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Finite Element Methods
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Paul J. Atzberger, UCSB

Finite Element Methods

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Application to Elliptic PDEs

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(RG-Approximation)

Motivations: Steady-state heat equation, electrostatics, incompressibility constraints.

Finite Element Approximation Steps:

i. Select element type for generating a space \( S \).

ii. Mesh the domain to obtain a collection of elements.
Application to Elliptic PDEs

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Application to Elliptic PDEs

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\end{align*}
\]

(RG-Approximation)

Motivations: Steady-state heat equation, electrostatics, incompressibility constraints.

Finite Element Approximation Steps:

1. Select element type for generating a space \( S \).
2. Mesh the domain to obtain a collection of elements.
3. Calculate the stiffness matrix and load vector using weak form.
4. Solve the linear system \( Ku = f \).
Application to Elliptic PDEs

Poisson Equation as Model Problem:

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

\[
\begin{align*}
\begin{cases}
\Delta u = -g, & x \in \Omega \\
u = f, & x \in \partial \Omega.
\end{cases}
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Discretization:

Paul J. Atzberger, UCSB

Finite Element Methods

http://atzberger.org/
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(RG-Approximation)

Discretization:
Divide domain into triangular elements \( T_j \).

\[
\begin{align*}
\Omega \\
\partial \Omega \\
f
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- Domain
- Triangulation
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Nodal basis \( \{ \phi_i \} \) are 2D "hat functions."
Functions in \( v \in S \) can be represented as

\[
v(x) = \sum_{i=1}^{n} v(x_i) \phi_i(x) \in H^1.
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Mesh Refinement:

- Can increase accuracy by refining the mesh.
- Many strategies possible.
- Here, edges of triangle are bisected.
- Recursively yields mesh refinements.
- Quality of the triangle shapes is important.
- Quality impacts condition number of the stiffness matrix \(K\).

Convergence expected sufficiently uniform refinements.
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Example:

Consider PDE with 
\( g(x, y) = \pi^2 \sin(\pi x) + \pi^2 \cos(\pi x) \) 
\( f(x, y) = \sin(\pi x) + \cos(\pi x) \).

Solution is 
\( u(x, y) = \sin(\pi x) + \cos(\pi x) \).

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Study the error vs mesh refinement \( N \sim h^{-2} \).

\[\text{Convergence Rate}\]
\[\text{error absolute vs number of elements}\]
\[\text{Log-log plots yield information on convergence rate}\]
\[\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h^{-r} + C) \Rightarrow -r/2 = s \sim -0.9\]
\[r \sim 1.8\]
Indicates 2nd-order convergence rate.
Need to develop theory to predict from element properties.

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Study the error vs mesh refinement \(N \sim h^{-2}\). Log-log plots yield information on convergence rate

\[\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h^r) + \log(C) \Rightarrow -r/2 = s \sim -0.9\]

Indicates 2nd-order convergence rate.

Need to develop theory to predict from element properties.

Paul J. Atzberger, UCSB

Finite Element Methods

http://atzberger.org/
Application to Elliptic PDEs

Poisson Equation as Model Problem:

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

\[
\begin{align*}
\begin{cases} 
\Delta u = -g, & x \in \Omega \\
u = f, & x \in \partial\Omega
\end{cases}
\end{align*}
\rightarrow \begin{cases} 
a(u, v) = -(g, v), & v \in S \\
a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx.
\end{cases}
\] (RG-Approximation)

Example:

Consider PDE with

\[
g(x, y) = \pi^2 \sin(\pi x) + \pi^2 \cos(\pi x) \\
f(x, y) = \sin(\pi x) + \cos(\pi x).
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Solution is

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