Sobolev Spaces

Paul J. Atzberger

206D: Finite Element Methods
University of California Santa Barbara
Basic Definitions

The $L^2(\Omega)$ for a smooth domain $\Omega$, denotes the space of all functions $f$ that are Lebegue square-integrable $\int_{\Omega} f^2 \, dx < \infty$. 

Definition: A function $u \in L^2$ has as its weak derivative $v = D^{\alpha} u = \partial^{\alpha} u$ if $(v, w)_{L^2} = (-1)^{|\alpha|} (u, \partial^{\alpha} w)_{L^2}$, $\forall w \in C_0^\infty$.

$C_\infty^\infty$ is the space of all functions is infinitely continuously differentiable. $C_\infty^\infty$ are all functions zero outside a compact set.
Basic Definitions

The $L^2(\Omega)$ for a smooth domain $\Omega$, denotes the space of all functions $f$ that are Lebesgue square-integrable $\int_\Omega f^2 \, dx < \infty$. We define the $L^2$-inner-product as

$$ (u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x) \, dx. $$

$C^\infty_0$ is the space of all functions that are infinitely continuously differentiable.

The $C^\infty_0 \subset C^\infty$ are all functions that are zero outside a compact set.
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This has the compatible $L^2$-norm

$$\|u\|_2 = \sqrt{(u, u)_{L^2}}.$$
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$C^\infty$ is the space of all functions is infinitely continuously differentiable. The $C_0^\infty \subset C^\infty$ are all functions zero outside a compact set.
For any integer $m \geq 0$, let $H^m$ be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order $m$, $|\alpha| \leq m$. 
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We define $k$-semi-norm as

$$|u|_k = \sqrt{\sum_{|\alpha| = k} (\partial^\alpha u, \partial^\alpha u)_0} = \sqrt{\sum_{|\alpha| = k} \|\partial^\alpha u\|^2_{L^2}}.$$
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We refer to $H^m$ with this inner-product as a **Sobolev space**. Also denoted by $W^{m, 2}$. 

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We can define Sobolev spaces without resorting directly to the notion of weak derivatives.
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**Theorem**

Let \( \Omega \subset \mathbb{R}^n \) be an open set with piecewise smooth boundary. Let \( m \geq 0 \), then \( C^\infty(\Omega) \cap H^m(\Omega) \) is dense in \( H^m(\Omega) \) under the norm \( \| \cdot \|_m \).
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This means that we can view $H^m$ as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $(\cdot, \cdot)_m$. 
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**Definition**

Denote the completion of $C^\infty_0(\Omega)$ under $\| \cdot \|_m$ by $H^m_0$. 
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**Definition**

Denote the completion of $C^\infty_0(\Omega)$ under $\| \cdot \|_m$ by $H^m_0(\Omega)$.

We have the following relations between the function spaces

\[
L^2(\Omega) = H^0(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \supset \cdots \supset H^m(\Omega)
\]

\[
= H^0_0(\Omega) \supset H^1_0(\Omega) \supset H^2_0(\Omega) \supset \cdots \supset H^m_0(\Omega).
\]
We can also define function spaces based on $L^p(\Omega)$, $C^\infty$, $C_0^\infty$ similarly using the norm $\| \cdot \|_p$. 

Definition

The Sobolev space denoted by $W^{m,p}$ (also by $W_0^{m,p}$) is the collection of functions obtained by completing $C_\infty(\Omega) \subset L^p(\Omega)$ under the norm $\| \cdot \|_m$.

Similarly, we obtain $W^{m,p}_0$ by completing $C_\infty^0(\Omega) \subset L^p(\Omega)$ under $\| \cdot \|_m$. 

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The Sobolev space denoted by $W^{m,p}$ (also by $W^m_p$) is the collection of functions obtained by completing $C^\infty(\Omega) \subset L^p(\Omega)$ under the norm $\| \cdot \|_m$.

Similarly, we obtain $W_0^{m,p}$ by completing $C_0^\infty(\Omega) \subset L^p(\Omega)$ under $\| \cdot \|_m$. 
Definition
Consider a given domain \( \Omega \) and compact sets \( K \subset \Omega \). We define the set of \textit{locally integrable} functions as

\[
L^1_{\text{loc}}(\Omega) := \{ v | v \in L^1(K), \forall K \subset \Omega^o \}
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Sobolev Spaces

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$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of $\Omega$ as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial \Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$. 

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Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| | x \in \Omega\}$$
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If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).
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Consider a given domain \( \Omega \) and compact sets \( K \subset \Omega \). We define the set of **locally integrable** functions as

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L_{\text{loc}}^1(\Omega) := \{ v | v \in L^1(K), \forall K \subset \Omega^o \}
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These functions can behave poorly near the boundary of \( \Omega \) as illustrated by \( v(x) = \phi(1/\text{dist}(x, \partial \Omega)) \) where \( \phi(x) = e^{e^x} \) which still yields \( v \in L_{\text{loc}}^1(\Omega) \).

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The \( p = \infty \) norm is given by

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If \( U = \text{ess-sup}(v) \) then \( v(x) \leq U \) for almost every \( x \in \Omega \) (except set of measure zero).

**Example:** Let \( f(x) = 3 \) on the rationals \( \mathbb{Q} \) and \( f(x) = 2 \) on the positive irrationals \( \mathbb{R}^+ \setminus \mathbb{Q} \) and
\( f(x) = -1 \) on the negative irrationals \( \mathbb{R}^- \setminus \mathbb{Q} \).
**Definition**

Consider a given domain $\Omega$ and compact sets $K \subset \Omega$. We define the set of *locally integrable* functions as

$$L^1_{\text{loc}}(\Omega) := \{ v | v \in L^1(K), \forall K \subset \Omega^o \}$$

These functions can behave poorly near the boundary of $\Omega$ as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial \Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

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The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

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*Example:* Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$. We have $\text{ess-sup}\{f(x) \mid x \in \Omega\} = 2$
Definition

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Definition

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**Definition**

For $1 \leq p < \infty$, we define the *Sobolev norm* as

$$
\| v \|_{W^k_p(\Omega)} := \left( \sum_{|\alpha| \leq k} \| D^\alpha w v \|_{L^p(\Omega)}^p \right)^{1/p},
$$

where $k$ is a non-negative integer, $v \in L^1_{\text{loc}}(\Omega)$, and $D^\alpha w v$ exists for all $|\alpha| \leq k$. For $p = \infty$, we define the Sobolev norm as

$$
\| v \|_{W^k_\infty(\Omega)} := \max_{|\alpha| \leq k} \| D^\alpha w v \|_{L^\infty(\Omega)}.
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Sobolev Spaces

**Definition**

For $1 \leq p < \infty$, we define the Sobolev norm as

$$\|v\|_{W^k_p(\Omega)} := \left( \sum_{|\alpha| \leq k} \|D_w^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

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Sobolev Spaces

**Definition**

The **Sobolev space** is defined as

\[ W_p^k(\Omega) := \{ v \in L^1_{\text{loc}}(\Omega) \mid \| v \|_{W_p^k(\Omega)} < \infty \} \]
Sobolev Spaces

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Definition

For \( 1 \leq p < \infty \), we define the **Sobolev semi-norm** as

\[ |v|_{W^k_p(\Omega)} := \left( \sum_{|\alpha| = k} \| D^\alpha_w v \|_{L^p(\Omega)}^p \right)^{1/p}, \]

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Sobolev Spaces

The general Sobolev spaces also satisfy inclusion relations.
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\textbf{Theorem}

For \( k, m \) are non-negative integers with \( k \leq m \) and \( p \) any real number with \( 1 \leq p \leq \infty \), we have

\[ W^m_p(\Omega) \subset W^k_p(\Omega). \]
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**Theorem**

For $k$ any non-negative integer and $p, q$ any real numbers with $1 \leq p \leq q \leq \infty$, we have

$$W^k_q(\Omega) \subset W^k_p(\Omega).$$
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For $k$ any non-negative integer and $p, q$ any real numbers with $1 \leq p \leq q \leq \infty$, we have

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**Theorem**

For $k, m$ non-negative integers with $k < m$ and and $p, q$ any real numbers with $1 \leq p < q \leq \infty$, we have

$$W_q^m(\Omega) \subset W_p^k(\Omega).$$
Poincaré-Friedrichs Inequality:

Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

$$\|v\|_0 \leq s|v|_1, \quad \forall v \in H^1_0(\Omega).$$
Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

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This shows the 1-semi-norm bounds the 0-norm.
Poincaré-Friedrichs Inequality

**Theorem**

**Poincaré-Friedrichs Inequality:** Consider domain $\Omega \subset Q = [0, s]^n$, $Q$ is cube of side-length $s$. Then

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**Proof:**
Poincaré-Friedrichs Inequality

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**Proof:** Since $v \in H^1_0$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express $v$ as

$$v(x_1, x_2, \ldots, x_n) = v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n)dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n)dz$$
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By the Cauchy-Swartz inequality we have

$$|v(x)|^2 \leq \left( \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 = \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$
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$$\leq s \int_0^{x_1} |\partial_1 v(z, x_2, \ldots, x_n)|^2 dz$$
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$$\leq s \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

We integrate over the cube $Q = [0, s]^n$ with $v$, $\partial^1 v$ extended to vanish outside of $\Omega$. 
Poincaré–Friedrichs Inequality

Theorem

**Poincaré–Friedrichs Inequality:** Consider the domain \( \Omega \subset [0, s]^n \) is contained within a cube of side-length \( s \). Then

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\|v\|_0 \leq s |v|_1, \quad \forall v \in H^1_0(\Omega).
\]

Proof:

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|v(x)|^2 \leq \left( \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 = \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \leq s \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
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Paul J. Atzberger, UCSB  
Finite Element Methods  
http://atzberger.org/
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$$\|v\|^2_0 = \|v\|^2 \leq s^2 \|\partial^1 v\|^2 = s^2|v|_1^2.$$

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**Theorem**

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When $\Omega$ is bounded, the $m$-semi-norm $|v|_m$ is in fact a proper norm on $H^m_0(\Omega)$. 
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The norm \( |v|_m \) is equivalent to \( \|v\|_m \) (convergence in one implies convergence in other).
Theorem

**Sobolev Inequality**: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with $k$ an integer, and $p$ real number with $1 \leq p < \infty$ such that

We then have there is a constant $C$ so that for all $u \in W^{k,p}(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}.$$

Also, for the equivalence class of $u$ in $L^\infty(\Omega)$, there is a representative that is a continuous function.

Significance:
- Shows that if a function has enough weak derivatives then in fact it can be viewed as equivalent to a continuous, bounded function.
- Also, shows that if we have convergence in $\| \cdot \|_{W^{k,p}(\Omega)}$ then also converges in $\| \cdot \|_{L^\infty(\Omega)}$. 
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Trace Theorems (boundary conditions)

When working with $L^p$ functions how do we characterize values on the boundary which are sets of measure zero.

**Example:** Consider $\Omega = \{(x, y) | x^2 + y^2 < 1\} = \{(r, \theta) | r < 1, 0 \leq \theta < 2\pi\}$.

**Lemma**

Let $\Omega$ be the unit disk. For all $u \in W^{1,2}(\Omega)$ the restriction of $u|_{\partial \Omega}$ can be interpreted as a function in $L^2(\partial \Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial \Omega)} \leq \frac{8}{14} \|u\|_{W^{1/2,2}(\Omega)}\|

**Proof (sketch):**

For $u \in C^1(\Omega)$, consider the restriction to $\partial \Omega$ when $r \leq 1$, $u(1, \theta)^2 = \int_0^1 \partial_r u(r, \theta)^2 r \, dr = \int_0^1 2r^2 u^2 \nabla u \cdot (x, y) r + ru^2) (r, \theta) \, dr \leq \int_0^1 2r^2 |u| |\nabla u| (r, \theta) \, dr$.
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\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|^{1/2}_{L^2(\Omega)} \|u\|^{1/2}_{W^{1,2}_0(\Omega)}.
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Paul J. Atzberger, UCSB

Finite Element Methods

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The norm of function $u|_{\partial \Omega}$ restricted to the boundary is

$$\|u\|_{L^2(\partial \Omega)}^2 := \int_{\partial \Omega} u^2 d\theta = \int_0^{2\pi} u(1, \theta)^2 d\theta.$$
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By Cauchy-Swartz we have

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\|u\|_{L^2(\partial \Omega)}^2 \leq 2\|u\|_{L^2(\Omega)} \left( \int_\Omega |\nabla u|^2 \, dx \, dy \right)^{1/2} + 2 \int_\Omega u^2 \, dx \, dy.
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Proof (sketch):

By Cauchy-Swartz we have

$$\|u\|^2_{L^2(\partial \Omega)} \leq 2\|u\|_{L^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + 2 \int_{\Omega} u^2 \, dx \, dy.$$

Using the arithmetic-geometric mean inequality we have

$$\left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + \left( \int_{\Omega} u^2 \, dx \, dy \right)^{1/2} \leq \left( 2 \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \, dy \right)^{1/2}.$$
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Let $\Omega$ be the unit disk. For all $u \in W^1_2(\Omega)$ the restriction of $u|_{\partial \Omega}$ can interpreted as a function in $L^2(\partial \Omega)$. Furthermore, it satisfies the bound

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Proof (sketch):

By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial \Omega)}^2 \leq 2 \|u\|_{L^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + 2 \int_{\Omega} u^2 \, dx \, dy.$$ 

Using the arithmetic-geometric mean inequality we have

$$\left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + \left( \int_{\Omega} u^2 \, dx \, dy \right)^{1/2} \leq 2 \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \, dy^{1/2}.$$ 

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Trace Theorems (boundary conditions)

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