Mixed Methods

Paul J. Atzberger

206D: Finite Element Methods
University of California Santa Barbara
We consider variational problems with constraints. Let $X$ and $M$ be Hilbert spaces with

- $a : X \times X \to \mathbb{R}$ (continuous bilinear forms)
- $b : X \times M \to \mathbb{R}$

**Saddle Point Problems**

Find the minimum $u \in X$ of

$$J[u] = \frac{1}{2} a(u, u) - (f, u)$$

subject to

$$b(u, \mu) = (g, \mu), \forall \mu \in M.$$

Consider the Lagrangian

$$L(u, \lambda) := J[u] + \left[ b(u, \lambda) - (g, \lambda) \right].$$

We seek the minimum of $L(\cdot, \lambda)$ with fixed $\lambda$.

Can we find $\lambda_0$ so this minimum satisfies the constraints?

When $L$ contains only bilinear and quadratic expressions in $u$ and $\lambda$, we obtain a saddle point problem.

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Saddle Point Problems

Saddle Point Problem I

Find \((u, \lambda) \in X \times M\) with

\[a(u, v) + b(v, \lambda) = \langle f, v \rangle, \forall v \in X,\]

\[b(u, v) = \langle g, \mu \rangle, \forall \mu \in M.\]

When the solution \((u^*, \lambda^*)\) is solution of the saddle-point conditions, this corresponds to

\[L(u^*, \lambda^*) \leq L(u^*, \lambda^*) \leq L(u, \lambda^*), \forall (u, \lambda) \in X \times M.\]

Assumes that \(a(v, v) \geq 0.\)

Solution in Infinite-Dimensional Spaces: we must not only have notion for definiteness of the bilinear form \(a\), but also of properties for the constraints \(b\) beyond simply linear independence.

Consider the overall linear mapping for the above problem \(L: X \times M \rightarrow X' \times M',\) maps \((u, \lambda) \mapsto (f, g).\)

Need ways to characterize when \(L\) is invertible (solvable) and the inverse is continuous (stable).
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Functional Analysis

Isomorphism

A linear mapping $L: U \rightarrow V$ with $U, V$ normed linear spaces is called an isomorphism if it is bijective and $L$ and $L^{-1}$ are continuous.

Consider a linear map associated with a bilinear form $a$ by $\langle Lu, v \rangle := a(u, v), \forall v \in V$.

Variational problem: $a(u, v) = \langle f, v \rangle, \forall v \in V \Rightarrow \langle Lu, v \rangle = \langle f, v \rangle$, formally $u = L^{-1}f$.

Definition: For $V \subset X$ closed, the $V^0 := \{ \ell \in X': \langle \ell, v \rangle = 0, \forall v \in V \}$ is called the polar set.

Theorem (Inf-Sup Condition) For Hilbert spaces $U, V$, the linear mapping $L: U \rightarrow V'$ is an isomorphism if and only if the corresponding bilinear form $a: U \times V \rightarrow \mathbb{R}$ satisfies the conditions:

(i) Continuity: There exists $C \geq 0$ so that $|a(u, v)| \leq C \|u\|_U \|v\|_V$.

(ii) Inf-Sup Condition: There exists $\alpha > 0$ such that $\inf_{u \in U} \sup_{v \in V} a(u, v) \|u\|_U \|v\|_V \geq \alpha > 0$.

(iii) For each $v \in V$, there exists $u \in U$ with $a(u, v) \neq 0$.
A linear mapping $L : U \to V$ with $U, V$ normed linear spaces is called an **isomorphism** if it is bijective and $L$ and $L^{-1}$ are continuous.

Consider a linear map associated with a bilinear form $a$ by $\langle L u, v \rangle := a(u, v), \forall v \in V$.

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Functional Analysis

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A linear mapping \( L : U \to V \) with \( U, V \) normed linear spaces is called an **isomorphism** if it is bijective and \( L \) and \( L^{-1} \) are continuous.

Consider a linear map associated with a bilinear form \( a \) by \( \langle Lu, v \rangle := a(u, v), \ \forall v \in V. \)

Variational problem: \( a(u, v) = \langle f, v \rangle, \ \forall v \in V \Rightarrow \langle Lu, v \rangle = \langle f, v \rangle, \) formally \( u = L^{-1}f. \)

Definition: For \( V \subset X \) closed, the \( V^0 := \{ \ell \in X' : \langle \ell, v \rangle = 0, \ \forall v \in V \} \) is called the **polar set**.

Theorem (Inf-Sup Condition)

For Hilbert spaces \( U, V \), the linear mapping \( L : U \to V' \) is an isomorphism if and only if the corresponding bilinear form \( a : U \times V \to \mathbb{R} \) satisfies the conditions:

(i) **Continuity**: There exists \( C \geq 0 \) so that \( |a(u, v)| \leq C \|u\|_U \|v\|_V. \)
(ii) **Inf-Sup Condition**: There exists \( \alpha > 0 \) such that

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Condition (i) readily implies the continuity of $L : U \rightarrow V'$.

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For $f \in L(U)$, by injectivity, exists unique $u = L^{-1}f$.

By (ii) $\Rightarrow \alpha \|u\|_U \leq \sup_{v \in V} a(u, v)\|v\|_V = \sup_{v \in V} \langle f, v \rangle\|v\|_V = \|f\|_{V'}$ $\Rightarrow \|Lu\|_{V'} \geq \alpha \|u\|_U \Rightarrow \|L^{-1}f\|_U \leq \frac{\alpha^{-1}}{\alpha} \|f\|_{V'}$,

so $L^{-1}$ is continuous on $\text{Im}(L)$.

Continuity of $L, L^{-1}$ implies $L(U)$ closed.

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Condition (i) readily implies the continuity of $L : U \to V'$. Condition (ii) gives us invertibility of $L$, since if $Lu_1 = Lu_2$ then $a(u_1, v) = a(u_2, v)$, $\forall v \in V$, giving $\sup_{v \in V} a(u_1 - u_2, v) = 0$. By (ii) this only occurs if $\|u_1 - u_2\|_U = 0$, so $u_1 = u_2$. For $f \in L(U)$, by injectivity, exists unique $u = L^{-1}f$.

By (ii) $\Rightarrow \alpha \|u\|_U \leq \sup_{v \in V} \frac{a(u, v)}{\|v\|_V} = \sup_{v \in V} \frac{\langle f, v \rangle}{\|v\|_V} = \|f\|_{V'} \Rightarrow ||Lu||_{V'} \geq \alpha \|u\|_U \Rightarrow \|L^{-1}f\|_U \leq \alpha^{-1} \|f\|_{V'}$, so $L^{-1}$ is continuous on $\text{Im}(L)$. Continuity of $L, L^{-1}$ implies $L(U)$ closed. Condition (iii) ensures only element in polar set is $\{0\}$ so $L$ is surjective (thm).
Theorem (Inf-Sup Condition)

For Hilbert spaces $U, V$, the linear mapping $L : U \to V'$ is an isomorphism if and only if the corresponding bilinear form $a : U \times V \to \mathbb{R}$ satisfies the conditions:

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\[
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\]

Lemma (Convergence): Consider $a: U \times V \to \mathbb{R}$ that satisfies the theorem based on Inf-Sup Conditions. Consider choosing approximation spaces $U_h \subset V, V_h \subset V$ for which the theorem also holds. Then
\[
\|u - u_h\| \leq 1 + C\alpha \inf_{w_h \in U_h} \|u - w_h\|.
\]

Remark: When this criteria holds for the spaces $U_h, V_h$, we say they satisfy the Babuska-Brezzi Condition.
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Consider bilinear form \( a : U \times V \to \mathbb{R} \) that satisfies the theorem based on Inf-Sup Conditions. Consider choice of approximation spaces \( U_h \subset V, V_h \subset V \) for which the theorem also holds. Then

\[
\| u - u_h \| \leq \left( 1 + \frac{C}{\alpha} \right) \inf_{w_h \in U_h} \| u - w_h \|.
\]

Proof:

For any \( w_h \in U_h \) we have

\[
a(u_h - w_h, v) = a(u - w_h, v), \quad \forall v \in V_h.
\]

Then, for \( \ell := a(u - w_h, \cdot) \), we have

\[
\| \ell \| \leq C \| u - w_h \|.
\]

By conditions (i)–(iii), the mapping \( L_h : U_h \to V_h' \) obtained from \( a(u_h - w_h, \cdot) \) satisfies \( \| L - L_h \| \leq \alpha - 1 \).

This gives

\[
\| u_h - w_h \| \leq \alpha - 1 \| \ell \| \leq \alpha - 1 C \| u - w_h \|.
\]

From triangle inequality,

\[
\| u - u_h \| \leq \| u - w_h \| + \| w_h - u_h \| \leq (1 + \alpha - 1 C) \| u - w_h \|.
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a(u - u_h, v) = 0, \quad \forall v \in V_h
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For \( \langle \ell, v \rangle := a(u - w_h, v) \), we have \( \|\ell\| \leq C\|u - w_h\| \).
**Lemma (Convergence)**

Consider bilinear form $a : U \times V \to \mathbb{R}$ that satisfies the theorem based on Inf-Sup Conditions. Consider choice of approximation spaces $U_h \subset V$, $V_h \subset V$ for which the theorem also holds. Then

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For $\langle \ell, v \rangle := a(u - w_h, v)$, we have $\| \ell \| \leq C \| u - w_h \|$. By conditions (i)–(iii), the mapping $L_h : U_h \to V'_h$ obtained from $a(u_h - w_h, \cdot)$ satisfies $\| L_h^{-1} \| \leq \alpha^{-1}$. 

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Finite Element Methods

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$$a(u - u_h, v) = 0, \quad \forall v \in V_h$$  

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$$a(u_h - w_h, v) = a(u - w_h, v), \quad \forall v \in V_h$$  

For $\langle \ell, v \rangle := a(u - w_h, v)$, we have $\|\ell\| \leq C\|u - w_h\|$. By conditions (i)–(iii), the mapping $L_h : U_h \to V'_h$ obtained from $a(u_h - w_h, \cdot)$ satisfies $\|L_h^{-1}\| \leq \alpha^{-1}$. This gives

$$\|u_h - w_h\| \leq \alpha^{-1}\|\ell\| \leq \alpha^{-1}C\|u - w_h\|.$$  

From triangle inequality,
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Consider bilinear form \( a : U \times V \to \mathbb{R} \) that satisfies the theorem based on Inf-Sup Conditions. Consider choice of approximation spaces \( U_h \subset V, V_h \subset V \) for which the theorem also holds. Then

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For \( \langle \ell, v \rangle := a(u - w_h, v) \), we have \( \|\ell\| \leq C\|u - w_h\| \). By conditions (i)–(iii), the mapping \( L_h : U_h \to V_h' \) obtained from \( a(u_h - w_h, \cdot) \) satisfies \( \|L_h^{-1}\| \leq \alpha^{-1} \). This gives

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\]

From triangle inequality,

\[
\|u - u_h\| \leq \|u - w_h\| + \|w_h - u_h\| \leq (1 + \alpha^{-1}C)\|u - w_h\|.
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For any \( w_h \in U_h \) we have

\[
a(u_h - w_h, v) = a(u - w_h, v), \forall v \in V_h
\]

For \( \langle \ell, v \rangle := a(u - w_h, v) \), we have \( \| \ell \| \leq C \| u - w_h \| \). By conditions (i)–(iii), the mapping \( L_h : U_h \to V_h' \) obtained from \( a(u_h - w_h, \cdot) \) satisfies \( \| L_h^{-1} \| \leq \alpha^{-1} \). This gives

\[
\| u_h - w_h \| \leq \alpha^{-1} \| \ell \| \leq \alpha^{-1} C \| u - w_h \|.
\]

From triangle inequality,

\[
\| u - u_h \| \leq \| u - w_h \| + \| w_h - u_h \| \leq (1 + \alpha^{-1} C) \| u - w_h \|.
\]
Saddle Point Problems

Returning to our original motivation.

Solution in Infinite-Dimensional Spaces:
Now have ways to characterize the properties of $a$, $b$ to ensure solution.
Consider the overall linear mapping for the above problem $L: X \times M \rightarrow X' \times M'$, maps $(u, \lambda) \mapsto (f, g)$.
We need to establish conditions for this to be an isomorphism.
Saddle Point Problems

Returning to our original motivation.

Saddle Point Problem I

\[ \begin{align*}
(\mathbf{u}, \lambda) \in X \times M \text{ with } & \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \lambda) = \langle f, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in X, \\
b(\mathbf{u}, \mu) = \langle g, \mu \rangle, \quad \forall \mu \in M.
\end{align*} \]

Solution in Infinite-Dimensional Spaces: Now have ways to characterize the properties of \(a, b\) to ensure solution.

Consider the overall linear mapping for the above problem \(L: X \times M \rightarrow X' \times M'\), maps \((\mathbf{u}, \lambda) \mapsto (f, g)\).

We need to establish conditions for this to be an isomorphism.
Returning to our original motivation.

**Saddle Point Problem I**

Find \((u, \lambda) \in X \times M\) with

\[
\begin{align*}
    a(u, v) + b(v, \lambda) &= \langle f, v \rangle, & \forall v \in X, \\
    b(u, \mu) &= \langle g, \mu \rangle, & \forall \mu \in M.
\end{align*}
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Saddle Point Problems

Returning to our original motivation.

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Saddle Point Problems

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Consider the overall linear mapping for the above problem.
Returning to our original motivation.

**Saddle Point Problem I**

Find \((u, \lambda) \in \mathcal{X} \times \mathcal{M}\) with

\[
\begin{align*}
    a(u, v) + b(v, \lambda) &= \langle f, v \rangle, \quad \forall v \in \mathcal{X}, \\
    b(u, \mu) &= \langle g, \mu \rangle, \quad \forall \mu \in \mathcal{M}.
\end{align*}
\]

**Solution in Infinite-Dimensional Spaces:** Now have ways to characterize the properties of \(a, b\) to ensure solution.

Consider the overall linear mapping for the above problem

\[
L : \mathcal{X} \times \mathcal{M} \to \mathcal{X}' \times \mathcal{M}', \quad \text{maps} \quad (u, \lambda) \mapsto (f, g).
\]
Returning to our original motivation.

### Saddle Point Problem I

Find \((u, \lambda) \in X \times M\) with

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\begin{align*}
    a(u, v) + b(v, \lambda) &= \langle f, v \rangle, & \forall v \in X, \\
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\[
L : X \times M \to X' \times M', \quad \text{maps} \quad (u, \lambda) \mapsto (f, g).
\]

We need to establish conditions for this to be an isomorphism.
Saddle Point Problems

Notation:

\[ V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \} \]

\[ V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \} \]

Since \( b \) is continuous, \( V \) is a closed subspace of \( X \).

Reformulation as an operator equation using bilinear form

\[ A : X \rightarrow X' \]
\[ \langle Au, v \rangle = a(u, v), \forall v \in X. \]

Similarly, for \( b(u, \cdot) \) we define \( B \) and adjoint \( B' \) as

\[ B : X \rightarrow M' \]
\[ B'(\lambda, v) = b(v, \lambda), \forall \lambda \in M, \forall v \in X. \]

The Saddle Point Problem I can be expressed as

Saddle Point Problem II

Find \((u, \lambda) \in X \times M\) satisfying

\[ Au + B'\lambda = f, \]
\[ Bu = g. \]
Saddle Point Problems

Notation: \( V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \} \),
Saddle Point Problems

**Notation:** $V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \}$, $V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \}$
Saddle Point Problems

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Saddle Point Problems

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Reformulation as an operator equation using bilinear form \( a(\cdot, \cdot) \)
**Saddle Point Problems**

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Since \( b \) is continuous, \( V \) is a closed subspace of \( X \).

Reformulation as an operator equation using bilinear form \( a(\cdot, \cdot) \)

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A : X \to X' \\
\langle Au, v \rangle = a(u, v), \quad \forall v \in X.
\]
Saddle Point Problems

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Saddle Point Problems

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Similarly, for \( b(\cdot, \cdot) \) we define \( B \) and adjoint \( B' \) as

\[
B : X \to M', \quad B' : M \to X' \\
\langle Bu, \mu \rangle = b(u, \mu), \forall \mu \in M, \quad \langle B' \lambda, v \rangle = b(v, \lambda), \forall v \in X.
\]
Saddle Point Problems

**Notation:** \( V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \} \), \( V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \} \)

Since \( b \) is continuous, \( V \) is a closed subspace of \( X \).

Reformulation as an operator equation using bilinear form \( a(\cdot, \cdot) \)

\[
A : X \to X' \quad \langle Au, v \rangle = a(u, v), \forall v \in X.
\]

Similarly, for \( b(\cdot, \cdot) \) we define \( B \) and adjoint \( B' \) as

\[
B : X \to M', \quad B' : M \to X' \quad \langle Bu, \mu \rangle = b(u, \mu), \forall \mu \in M, \langle B' \lambda, v \rangle = b(v, \lambda), \forall v \in X.
\]

The Saddle Point Problem I can be expressed as
Saddle Point Problems

**Notation:** $V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \ \forall \mu \in M \}, \ V := \{ v \in X : b(v, \mu) = 0, \ \forall \mu \in M \}$

Since $b$ is continuous, $V$ is a closed subspace of $X$.

Reformulation as an operator equation using bilinear form $a(\cdot, \cdot)$

$$
A : X \to X', \quad \langle Au, v \rangle = a(u, v), \ \forall v \in X.
$$

Similarly, for $b(\cdot, \cdot)$ we define $B$ and adjoint $B'$ as

$$
B : X \to M', \quad B' : M \to X', \quad \langle Bu, \mu \rangle = b(u, \mu), \ \forall \mu \in M, \quad \langle B' \lambda, v \rangle = b(\nu, \lambda), \ \forall v \in X.
$$

The Saddle Point Problem I can be expressed as

**Saddle Point Problem II**
**Saddle Point Problems**

**Notation:**\[ V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \}, \quad V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \} \]

Since \( b \) is continuous, \( V \) is a closed subspace of \( X \).

Reformulation as an operator equation using bilinear form \( a(\cdot, \cdot) \)

\[
A : X \to X', \quad \langle Au, v \rangle = a(u, v), \quad \forall v \in X.
\]

Similarly, for \( b(\cdot, \cdot) \) we define \( B \) and adjoint \( B' \) as

\[
B : X \to M', \quad B' : M \to X', \quad \langle Bu, \mu \rangle = b(u, \mu), \quad \forall \mu \in M, \quad \langle B'\lambda, v \rangle = b(\nu, \lambda), \quad \forall v \in X.
\]

The Saddle Point Problem I can be expressed as

**Saddle Point Problem II**

Find \((u, \lambda) \in X \times M\) satisfying
Saddle Point Problems

**Notation:** \( V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \} \), \( V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \} \)

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\]

The Saddle Point Problem I can be expressed as

**Saddle Point Problem II**

Find \((u, \lambda) \in X \times M\) satisfying

\[
Au + B' \lambda = f, \\
Bu = g.
\]
Saddle Point Problems

**Notation:** \( V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \ \forall \mu \in M \} \), \( V := \{ v \in X : b(v, \mu) = 0, \ \forall \mu \in M \} \)

Since \( b \) is continuous, \( V \) is a closed subspace of \( X \).

Reformulation as an operator equation using bilinear form \( a(\cdot, \cdot) \)

\[
A : X \rightarrow X' \\
\langle Au, v \rangle = a(u, v), \ \forall v \in X.
\]

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B : X \rightarrow M' , \quad B' : M \rightarrow X' \\
\langle Bu, \mu \rangle = b(u, \mu), \ \forall \mu \in M, \quad \langle B' \lambda, v \rangle = b(v, \lambda), \ \forall v \in X.
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The Saddle Point Problem I can be expressed as

**Saddle Point Problem II**

Find \((u, \lambda) \in X \times M\) satisfying

\[
Au + B' \lambda = f, \\
Bu = g.
\]
Saddle Point Problems

Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} b(v, \mu) \|v\| \|\mu\| \geq \beta > 0 \).

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \forall v \in V^\perp \).

(iii) The operator \( B' : M \to V_0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \forall \mu \in M \).

Proof: The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.

We show (iii) \( \Rightarrow \) (ii).

For \( v \in V^\perp \) let \( g \in V_0 \) defined by mapping \( w \mapsto (v, w) \).

By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that \( b(w, \lambda) = (v, w), \forall w \in V \).

From the definition of the functional \( \|g\| = \|v\| \).

Also, \( \|B'\mu\| \geq \beta \|\mu\| \) so \( \|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\| \).

Substituting into \( b(\cdot, \cdot) \) above \( w = v \), we have

\[ \sup_{\mu \in M} b(v, \mu) \|\mu\| \geq b(v, \mu) \|\mu\| = (v, v) \|\lambda\| \geq \beta \|v\|. \]

The \( B : V^\perp \to M' \) satisfies the conditions of Inf-Sup Lemma so the mapping \( B \) is an isomorphism.

Therefore, (iii) \( \Rightarrow \) (ii).
Saddle Point Problems

Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} b(v, \mu) \|v\| \|\mu\| \geq \beta > 0 \).

(ii) The operator \( B : V^\perp \rightarrow M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \forall v \in V^\perp \).

(iii) The operator \( B' : M \rightarrow V_0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \forall \mu \in M \).

Proof: The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma. We show (iii) \( \Rightarrow \) (ii).

For \( v \in V^\perp \) let \( g \in V_0 \) defined by mapping \( w \mapsto (v, w) \).

By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that \( b(w, \lambda) = (v, w), \forall w \in V \).

From the definition of the functional \( \|g\| = \|v\| \).

Also, \( \|B'\mu\| \geq \beta \|\mu\| \) so \( \|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\| \).

Substituting into \( b(\cdot, \cdot) \) above \( w = v \), we have \( \sup_{\mu \in M} \ b(v, \mu) \|\mu\| \geq b(v, \mu) \|\mu\| = (v, v) \|\lambda\| \geq \beta \|v\| \).

The \( B : V^\perp \rightarrow M' \) satisfies the conditions of Inf-Sup Lemma so the mapping \( B \) is an isomorphism.

Therefore, (iii) \( \Rightarrow \) (ii).
Inf-Sup Lemma

The following conditions are equivalent

(i) $\inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0$.

(ii) The operator $B : V^\perp \rightarrow M'$ is an isomorphism and $\|Bv\| \geq \beta \|v\|$, $\forall v \in V^\perp$.

(iii) The operator $B' : M \rightarrow V_0 \subset X'$ is an isomorphism and $\|B'\mu\| \geq \beta \|\mu\|$, $\forall \mu \in M$.

Proof: The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma. We show (iii) $\Rightarrow$ (ii).

For $v \in V^\perp$ let $g \in V_0$ defined by mapping $w \mapsto (v, w)$.

By (iii) $B'$ is an isomorphism so there exists $\lambda \in M$ so that $b(w, \lambda) = (v, w)$, $\forall w \in V$.

From the definition of the functional $\|g\| = \|v\|$.

Also, $\|B'\mu\| \geq \beta \|\mu\|$ so $\|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\|$.

Substituting into $b(\cdot, \cdot)$ above $w = v$, we have $\sup_{\mu \in M} b(v, \mu) \|\mu\| \geq b(v, \mu) \|\mu\| = (v, v) \|\lambda\| \geq \beta \|v\|$.

The $B : V^\perp \rightarrow M'$ satisfies the conditions of Inf-Sup Lemma so the mapping $B$ is an isomorphism.

Therefore, (iii) $\Rightarrow$ (ii).
The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

(ii) The operator \( B : V^\perp \rightarrow M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \forall v \in V^\perp. \)

Proof: The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma. We show (iii) \( \Rightarrow \) (ii).

For \( v \in V^\perp \) let \( g \in V_0 \) defined by mapping \( w \mapsto (v, w) \).

By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that \( b(w, \lambda) = (v, w), \forall w \in V \).

From the definition of the functional \( \|g\| = \|v\| \).

Also, \( \|B'\mu\| \geq \beta \|\mu\| \) so \( \|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\| \).

Substituting into \( b(\cdot, \cdot) \) above \( w = v \), we have

\[ \sup_{\mu \in M} b(v, \mu) \quad \|\mu\| \geq b(v, \mu) \quad \|\mu\| = (v, v) \quad \|\lambda\| \geq \beta \|v\|. \]

Therefore, (iii) \( \Rightarrow \) (ii).
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0 \).

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \ \forall v \in V^\perp \).

(iii) The operator \( B' : M \to V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \ \forall \mu \in M \).

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
We show (iii) \( \Rightarrow \) (ii).
For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w) \).
By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that \( b(w, \lambda) = (v, w), \ \forall w \in V^\perp \).
From the definition of the functional \( \|g\| = \|v\| \).
Also, \( \|B'\mu\| \geq \beta \|\mu\| \) so \( \|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\| \).
Substituting into \( b(\cdot, \cdot) \) above \( w = v \), we have
\[ \sup_{\mu \in M} b(v, \mu) \geq \frac{b(v, \mu)}{\|\mu\|} \geq \beta \|v\| = (v, v) \\|\lambda\| \geq \beta \|v\|. \]

The \( B : V^\perp \to M' \) satisfies the conditions of Inf-Sup Lemma so the mapping \( B \) is an isomorphism.

Therefore, (iii) \( \Rightarrow \) (ii).
Saddle Point Problems

**Inf-Sup Lemma**

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

(ii) The operator \( B : V^\perp \rightarrow M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \ \forall v \in V^\perp. \)

(iii) The operator \( B' : M \rightarrow V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \ \forall \mu \in M. \)

**Proof:**
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

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(iii) The operator \( B' : M \rightarrow V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \quad \forall \mu \in M. \)

**Proof:**

The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\|\|\mu\|} \geq \beta > 0 \).

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \ \forall v \in V^\perp \).

(iii) The operator \( B' : M \to V^0 \subset X' \) is an isomorphism and \( \|B' \mu\| \geq \beta \|\mu\|, \ \forall \mu \in M \).

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma. We show (iii) \( \Rightarrow \) (ii).
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

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(iii) The operator \( B' : M \to V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \forall \mu \in M. \)

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma. We show (iii) \( \Rightarrow \) (ii). For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w) \).
Saddle Point Problems

Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

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(iii) The operator \( B' : M \rightarrow V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \forall \mu \in M. \)

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
We show (iii) \( \Rightarrow \) (ii). For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w) \). By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that...
The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \forall v \in V^\perp. \)

(iii) The operator \( B' : M \to V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \forall \mu \in M. \)

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
We show (iii) \( \Rightarrow \) (ii). For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w) \). By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that

\[ b(w, \lambda) = (v, w), \forall w \in V. \]
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \; \forall v \in V^\perp. \)

(iii) The operator \( B' : M \to V^0 \subset X' \) is an isomorphism and \( \|B'\mu\| \geq \beta \|\mu\|, \; \forall \mu \in M. \)

Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
We show (iii) \( \Rightarrow \) (ii). For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w) \). By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that

\[ b(w, \lambda) = (v, w), \; \forall w \in V. \]

From the definition of the functional \( \|g\| = \|v\|. \)
### Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0 \).

(ii) The operator \( B : V^\perp \to M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \forall v \in V^\perp \).

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The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.

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\forall w \in V, \quad b(w, \lambda) = (v, w).
\]

From the definition of the functional \( \|g\| = \|v\| \). Also, \( \|B'\mu\| \geq \beta \|\mu\| \) so \( \|v\| = \|g\| = \|B'\lambda\| \geq \beta \|\lambda\| \).
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Substituting into \( b(\cdot, \cdot) \) above \( w = v \), we have
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Substituting into \( b(\cdot, \cdot) \) above \( w = v, \) we have

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    \sup_{\mu \in M} \frac{b(v, \mu)}{\|\mu\|} \geq \frac{b(v, \mu)}{\|\mu\|} = \frac{(v, v)}{\|\lambda\|} \geq \beta \|v\|.
\]
Inf-Sup Lemma

The following conditions are equivalent
(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)
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Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
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Substituting into \( b(\cdot, \cdot) \) above \( w = v, \) we have
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\sup_{\mu \in M} \frac{b(v, \mu)}{\|\mu\|} \geq \frac{b(v, \mu)}{\|\mu\|} = \frac{(v, v)}{\|\lambda\|} \geq \beta \|v\|.
\]
The \( B : V^\perp \to M' \) satisfies the conditions of Inf-Sup Lemma so the mapping \( B \) is an isomorphism.
## Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0. \)

(ii) The operator \( B : V^\perp \rightarrow M' \) is an isomorphism and \( \|Bv\| \geq \beta \|v\|, \ \forall v \in V^\perp. \)

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**Proof:**

The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.

We show (iii) \( \Rightarrow \) (ii). For \( v \in V^\perp \) let \( g \in V^0 \) defined by mapping \( w \mapsto (v, w). \) By (iii) \( B' \) is an isomorphism so there exists \( \lambda \in M \) so that

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Substituting into \( b(\cdot, \cdot) \) above \( w = v, \) we have

\[ \sup_{\mu \in M} \frac{b(v, \mu)}{\|\mu\|} \geq \frac{b(v, \mu)}{\|\mu\|} = \frac{(v, v)}{\|\lambda\|} \geq \beta \|v\|. \]

The \( B : V^\perp \rightarrow M' \) satisfies the conditions of Inf-Sup Lemma so the mapping \( B \) is an isomorphism. Therefore, (iii) \( \Rightarrow \) (ii).
Inf-Sup Lemma

The following conditions are equivalent

(i) \( \inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0 \).

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Proof:

We show (ii) \(\Rightarrow\) (i).
Saddle Point Problems

Inf-Sup Lemma

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We show (ii) \( \Rightarrow \) (i). By (ii), \( B : V^\perp \to M' \) is an isomorphism.
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\]
**Inf-Sup Lemma**

The following conditions are equivalent

(i) $\inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\|v\| \|\mu\|} \geq \beta > 0$.

(ii) The operator $B : V^\perp \rightarrow M'$ is an isomorphism and $\|Bv\| \geq \beta \|v\|$, $\forall v \in V^\perp$.

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\]

Therefore, (ii) \( \Rightarrow \) (i).
\[\blacksquare\]
Notation:

\[ V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \forall \mu \in M \} \]

\[ V := \{ v \in X : b(v, \mu) = 0, \forall \mu \in M \} \]

A central theorem for saddle point problems.

Brezzi's Splitting Theorem

For the Saddle Point Problem I, the mapping \( L \) is an isomorphism \( L : X \times M \to X' \times M' \) if and only if the following two conditions are satisfied

1. The bilinear form \( a(\cdot, \cdot) \) is elliptic (coercive) in \( V \),
   \[ a(v, v) \geq \alpha \|v\|^2, \forall v \in V \text{ with } \alpha > 0, \]
   \( V \) given above.

2. The bilinear form \( b(\cdot, \cdot) \) satisfies the inf-sup condition
   \[ \inf_{\mu \in M} \sup_{v \in X} b(v, \mu) \|v\| \|\mu\| \geq \beta. \]

Remark:
Note the coercivity is assumed only for \( v \) in kernel of \( B \) (see def. of \( V \)).

Provides conditions directly in terms of the bilinear forms \( a \) and \( b \) concerning solvability.

Referred to as the Brezzi Conditions or Ladyzhenskaya-Babuska-Brezzi (LBB-Conditions).
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Saddle Point Problems

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Saddle Point Problems

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Saddle Point Problems

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**Saddle Point Problems**

**Notation:** \( V(g) := \{ v \in X : b(v, \mu) = \langle g, \mu \rangle, \ \forall \mu \in M \} \), \( V := \{ v \in X : b(v, \mu) = 0, \ \forall \mu \in M \} \)

A central theorem for saddle point problems.

**Brezzi’s Splitting Theorem**

For the *Saddle Point Problem I*, the mapping \( L : X \times M \to X' \times M' \) if and only if the following two conditions are satisfied

(i) The bilinear form \( a(\cdot, \cdot) \) is elliptic (coercive) in \( V \), \( a(v, v) \geq \alpha \| v \|^2, \ \forall v \in V \) with \( \alpha > 0 \), \( V \) given above.

(ii) The bilinear form \( b(\cdot, \cdot) \) satisfies the inf-sup condition

\[
\inf_{\mu \in M} \sup_{v \in X} \frac{b(v, \mu)}{\| v \| \| \mu \|} \geq \beta.
\]

**Remark:** Note the coercivity is assumed only for \( v \) in kernel of \( B \) (see def. of \( V \)).

Provides conditions directly in terms of the bilinear forms \( a \) and \( b \) concerning solvability.

Referred to as the Brezzi Conditions or Ladyzhenskaya-Babuska-Brezzi (LBB-Conditions).
Find $(u_h, \lambda_h) \in X_h \times M_h$ so that

$$a(u_h, v) + b(v, \lambda_h) = \langle f, v \rangle, \quad \forall v \in X_h$$

$$b(u_h, \mu) = \langle g, \mu \rangle, \quad \forall \mu \in M_h.$$  

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We say the Babuska-Brezzi Condition is satisfied by a family of finite element spaces $X_h, M_h$ if there exists $\alpha > 0, \beta > 0$ independent of $h$ so that

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Remark: Also referred to as the Inf-Sup Conditions.
Mixed Finite Element Methods

Mixed FEM I

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Paul J. Atzberger, UCSB

http://atzberger.org/
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Mixed Finite Element Methods

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Mixed Methods

Theorem

When $X_h$ and $M_h$ satisfy the Babuska-Brezzi conditions (and assumptions of the prior theorem), then

\[ \parallel u - u_h \parallel + \parallel \lambda - \lambda_h \parallel \leq c \inf_{v_h \in X_h} \parallel u - v_h \parallel + \inf_{\mu_h \in M_h} \parallel \lambda - \mu_h \parallel \]

Remark: Generally, $V_h \not\subset V$ (non-conforming). We usually do get better results in conforming case $V_h \subset V$.

Definition: The spaces $X_h \subset X$ and $M_h \subset M$, are said to satisfy condition (C) provided $V_h \subset V$.

Significance: Condition (C) $\Rightarrow \forall v_h \in X_h, b(v_h, \mu_h) = 0$, $\forall \mu_h \in M_h \Rightarrow b(v_h, \mu) = 0$, $\forall \mu \in M$.

Theorem: Suppose assumptions of prior theorem and Condition (C) satisfied. The solution to Mixed FEM I satisfies

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Mixed Methods

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When $X_h$ and $M_h$ satisfy the Babuska-Brezzi conditions (and assumptions of the prior theorem), then

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\[ \| u - u_h \| \leq c \inf_{v_h \in X_h} \| u - v_h \|. \]

Proof:
Consider \( v_h \in V_h(\mathbf{g}) \). It follows that

\[ a(u_h - v_h, v_h) = a(u_h, v_h) - a(u, v) + a(u - v_h, v_h) \]

\[ = b(v, \lambda - \lambda_h) + a(u - v_h, v_h) \leq C \| u - v_h \| \cdot \| v \|. \]

Holds \( \forall v \in V_h \) since \( b(v, \lambda - \lambda_h) = 0 \) from Condition (C).

Let \( v := u_h - v_h \), then

\[ \| u_h - v_h \|_2 \leq \alpha^{-1} C \| u_h - v_h \| \cdot \| u - v_h \|. \]

Dividing by \( \| u_h - v_h \| \), we have

\[ \| u_h - v_h \| \leq \alpha^{-1} C \| u - v_h \|. \]

By triangle inequality,

\[ \| u - u_h \| \leq \| u - v_h \| + \| v_h - u_h \| \]

and the result follows.

\[ \square \]
Mixed Methods

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Suppose assumptions of prior theorem and \textit{Condition (C)} satisfied. The solution to Mixed FEM I satisfies

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Proof:

Consider \(v_h \in V_h(g)\). It follows that

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Let \(v := u_h - v_h\), then

\[\|u_h - v_h\| \leq \alpha - 1 C \|u - v_h\| \cdot \|u - v_h\|.\]

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\[ \|u - u_h\| \leq c \inf_{v_h \in X_h} \|u - v_h\|. \]

Proof:

Consider \( v_h \in V_h \). It follows that

\[ a(u_h - v_h, v) = a(u_h, v) - a(u, v) + a(u - v_h, v) = b(v, \lambda - \lambda_h) + a(u - v_h, v) \leq C \|u - v_h\| \|v\|. \]

Holds \( \forall v \in V_h \) since \( b(v, \lambda - \lambda_h) = 0 \) from Condition (C).

Let \( v := u_h - v_h \), then

\[ \|u_h - v_h\| \leq \alpha - 1 C \|u_h - v_h\| \|u - v_h\|. \]

Dividing by \( \|u_h - v_h\| \), we have

\[ \|u_h - v_h\| \leq \alpha - 1 C \|u - v_h\|. \]

By triangle inequality, \( \|u - u_h\| \leq \|u - v_h\| + \|v_h - u_h\| \) and the result follows. ■

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Finite Element Methods

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Consider \(v_h \in V_h(g)\). It follows that

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\]

Dividing by \(\|u - u_h\|\), we have

\[
\|u - u_h\| \leq c \alpha - 1 \|u - v_h\|.
\]

By triangle inequality,

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\(\blacksquare\)
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\[ a(u_h - v_h, v) = a(u_h, v) - a(u, v) + a(u - v_h, v) \]
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\[
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Mixed Methods

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Let \( v := u_h - v_h \), then

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Suppose assumptions of prior theorem and \textit{Condition (C)} satisfied. The solution to Mixed FEM I satisfies

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Mixed Methods

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\[
\begin{align*}
a(u_h - v_h, v) & = a(u_h, v) - a(u, v) + a(u - v_h, v) \\
& = b(v, \lambda - \lambda_h) + a(u - v_h, v) \\
& \leq C \| u - v_h \| \cdot \| v \|.
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Mixed Methods

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\[ \blacksquare \]
Poisson Problem: Mixed Methods

Poisson Problem:

\[ \Delta u = -f, \quad x \in \Omega, \]
\[ u = 0, \quad x \in \Gamma_0, \]
\[ \nabla u \cdot n = 0, \quad x \in \Gamma_1. \]

We use that \( \Delta u = \text{div} \text{ grad} u \).

Let \( \sigma = \text{grad} u \),

then the Poisson problem becomes

\[ \text{grad} u = \sigma, \quad \text{div} \sigma = -f. \]

Poisson Problem: Mixed Formulation

Find \((\sigma, u) \in L^2(\Omega) \times H^1_0(\Omega)\) so that

\[ (\sigma, \tau)_0, \Omega - (\tau, \nabla u)_0, \Omega = 0, \quad \forall \tau \in L^2(\Omega), \]
\[ (\sigma, \nabla v)_0, \Omega = -(f, v)_0, \Omega, \quad \forall v \in H^1_0(\Omega). \]
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\[ (\sigma, \tau)_0, \Omega \] \[ - (\tau, \nabla u)_0, \Omega \] \[ = 0, \ \forall \tau \in L^2(\Omega) \] \[ - (\sigma, \nabla v)_0, \Omega \] \[ = - (f, v)_0, \Omega, \ \forall v \in H^1_0(\Omega) \].
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\]

\[
(\sigma, \nabla v)_{L^2(\Omega)} = - (f, v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega).
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We use that \( \Delta u = \text{div} \, \text{grad} \, u \). Let \( \sigma = \text{grad} \, u \), then the Poisson problem becomes

\[
\begin{align*}
\text{grad} \, u &= \sigma \\
\text{div} \, \sigma &= -f
\end{align*}
\]
Poisson Problem: Mixed Methods

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Poisson Problem: Mixed Methods

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Find \((\sigma, u) \in L_2(\Omega)^d \times H^1_0(\Omega)\) so that
\[
(\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} = 0, \quad \forall \tau \in L_2(\Omega)^d
\]
Poisson Problem: Mixed Methods

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Find \((\sigma, u) \in L_2(\Omega)^d \times H_0^1(\Omega)\) so that

\[
\begin{align*}
(\sigma, \tau)_{0, \Omega} - (\tau, \nabla u)_{0, \Omega} &= 0, \ \forall \tau \in L_2(\Omega)^d \\
-(\sigma, \nabla v)_{0, \Omega} &= -(f, v)_{0, \Omega}, \ \forall v \in H_0^1(\Omega).
\end{align*}
\]
Poisson Problem: Mixed Methods

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\[ \Delta u = -f, \quad x \in \Omega, \quad u = 0, \quad x \in \Gamma_0, \quad \nabla u \cdot \mathbf{n} = 0, \quad x \in \Gamma_1. \]

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(\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} &= 0, \quad \forall \tau \in L_2(\Omega)^d \\
-(\sigma, \nabla \nu)_{0,\Omega} &= -(f, \nu)_{0,\Omega}, \quad \forall \nu \in H_0^1(\Omega).
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Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

Find \((\sigma, u) \in L^2(\Omega) \times H^1_0(\Omega)\) so that

\[
(\sigma, \tau) = (\sigma, \tau)_{0, \Omega},
\]

\[
\tau, \nabla u = 0, \forall \tau \in L^2(\Omega),
\]

\[
(\sigma, \nabla v) = -\langle f, v \rangle_{0, \Omega}, \forall v \in H^1_0(\Omega).
\]

Poisson Problem: Saddle-Point Formulation

Let \(X := L^2(\Omega)\), \(M := H^1_0(\Omega)\)

\[
a(\sigma, \tau) := (\sigma, \tau)_{0, \Omega},
\]

\[
b(\tau, v) := -(\tau, \nabla v)_{0, \Omega}.
\]

Saddle-Point Problem:

\[
a(\sigma, \tau) - b(\tau, v) = 0
\]

\[
b(\sigma, \tau) = -\langle f, v \rangle_{0, \Omega}.
\]
Find \((\sigma, u) \in L_2(\Omega)^d \times H^1_0(\Omega)\) so that

\[
\begin{align*}
(\sigma, \tau) &\quad_{\Omega} - (\tau, \nabla u)_{\Omega} = 0, \\
(\sigma, \nabla v)_{\Omega} &\quad_{\Omega} = -\langle f, v \rangle_{\Omega},
\end{align*}
\]

Poisson Problem: Saddle-Point Formulation

Let \(X := L_2(\Omega)^d\), \(M := H^1_0(\Omega)\)

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Saddle-Point Problem:

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a(\sigma, \tau) - b(\tau, v) = 0
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Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

Find \((\sigma, u) \in L_2(\Omega)^d \times H^1_0(\Omega)\) so that

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\]

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Poisson Problem: Saddle-Point Formulation

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a(\sigma, \tau) - b(\tau, v) = 0
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Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

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Poisson Problem: Saddle-Point Formulation

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\]

Saddle-Point Problem:
Poisson Problem: Mixed Methods

### Poisson Problem: Mixed Formulation

Find \((\sigma, u) \in L^2(\Omega)^d \times H^1_0(\Omega)\) so that

\[
\begin{align*}
(\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} &= 0, \quad \forall \tau \in L^2(\Omega)^d, \\
-(\sigma, \nabla v)_{0,\Omega} &= -(f, v)_{0,\Omega}, \quad \forall v \in H^1_0(\Omega).
\end{align*}
\]

### Poisson Problem: Saddle-Point Formulation

Let

\[ X := L^2(\Omega)^d, \quad M := H^1_0(\Omega) \]
\[ a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}. \]

Saddle-Point Problem:

\[ a(\sigma, \tau) - b(\tau, v) = 0 \]
Poisson Problem: Mixed Formulation

Find \((\sigma, u) \in L_2(\Omega)^d \times H_0^1(\Omega)\) so that

\[
\begin{align*}
(\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} & = 0, \quad \forall \tau \in L_2(\Omega)^d \\
-(\sigma, \nabla v)_{0,\Omega} & = -(f, v)_{0,\Omega}, \quad \forall v \in H_0^1(\Omega).
\end{align*}
\]

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a(\sigma, \tau) - b(\tau, v) & = 0 \\
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Poisson Problem: Mixed Methods

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Saddle-Point Problem:

\[ a(\sigma, \tau) - b(\tau, v) = 0 \]

\[ b(\sigma, \tau) = -\langle f, v \rangle_{0, \Omega}. \]
Poisson Problem: Mixed Methods

Let

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\[ a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}. \]

Saddle-Point Problem:

\[ a(\sigma, \tau) - b(\tau, v) = 0 \]

\[ b(\sigma, \tau) = -\langle f, v \rangle_{0,\Omega}. \]

The Inf-Sup Condition holds since

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### Poisson Problem: Mixed Methods

Let

\[
X := L_2(\Omega)^d, \quad M := H^1_0(\Omega)
\]

\[
a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}.
\]

#### Saddle-Point Problem:

\[
a(\sigma, \tau) - b(\tau, v) = 0
\]

\[
b(\sigma, \tau) = -\langle f, v \rangle_{0,\Omega}.
\]

The Inf-Sup Condition holds since

\[
\frac{b(\tau, v)}{\|\tau\|_0} \geq 1
\]

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Poisson Problem: Mixed Methods

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\[ X := L_2(\Omega)^d, \quad M := H^1_0(\Omega) \]

\[ a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := - (\tau, \nabla v)_{0,\Omega}. \]

Saddle-Point Problem:

\[ a(\sigma, \tau) - b(\tau, v) = 0 \]

\[ b(\sigma, \tau) = - \langle f, v \rangle_{0,\Omega}. \]

The Inf-Sup Condition holds since

\[ \frac{b(\tau, v)}{\|\tau\|_0} = - \frac{(\tau, \nabla v)_{0,\Omega}}{\|\tau\|_0} \]

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Poisson Problem: Mixed Methods

Let

\[ X := L_2(\Omega)^d, \ M := H_0^1(\Omega) \]

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Saddle-Point Problem:

\[ a(\sigma, \tau) - b(\tau, v) = 0 \]
\[ b(\sigma, \tau) = - \langle f, v \rangle_{0,\Omega}. \]

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\[ \frac{b(\tau, v)}{\|\tau\|_0} = \frac{-(\tau, \nabla v)_{0,\Omega}}{\|\tau\|_0} \rightarrow \frac{(\nabla v, \nabla v)_{0,\Omega}}{\|\nabla v\|_0} \]
Poisson Problem: Mixed Methods

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\[ X := L_2(\Omega)^d, \quad M := H^1_0(\Omega) \]

\[ a(\sigma, \tau) := \langle \sigma, \tau \rangle_{0, \Omega}, \quad b(\tau, v) := -\langle \tau, \nabla v \rangle_{0, \Omega}. \]

Saddle-Point Problem:

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\[ b(\sigma, \tau) = -\langle f, v \rangle_{0, \Omega}. \]

The Inf-Sup Condition holds since

\[ \frac{b(\tau, v)}{\|\tau\|_0} = -\frac{\langle \tau, \nabla v \rangle_{0, \Omega}}{\|\tau\|_0} \rightarrow \frac{\langle \nabla v, \nabla v \rangle_{0, \Omega}}{\|\nabla v\|_0} = |v|_1 \]
Poisson Problem: Mixed Methods

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Poisson Problem: Mixed Methods

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This establishes stability of the formulation.
Poisson Problem: Mixed Methods

Poisson Problem: Saddle-Point Formulation

\[ X := L_2(\Omega)^d, \quad M := H_0^1(\Omega), \quad a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -\langle \tau, \nabla v \rangle_{0,\Omega}. \]

Saddle-Point Problem:

\[
\begin{align*}
  a(\sigma, \tau) - b(\tau, v) &= 0 \\
  b(\sigma, \tau) &= -\langle f, v \rangle_{0,\Omega}.
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Poisson Problem: Mixed Methods

Poisson Problem: Saddle-Point Formulation

\[ X := L^2(\Omega)^d, \quad M := H^1_0(\Omega), \quad a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}. \]

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We can obtain stable Finite Element discretizations for triangulations \( \mathcal{T}_h \).
Poisson Problem: Mixed Methods

Poisson Problem: Saddle-Point Formulation

\[ X := L_2(\Omega)^d, \quad M := H_0^1(\Omega), \quad a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}. \]

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We can obtain stable Finite Element discretizations for triangulations \( T_h \). For \( k \geq 1 \), let
Poisson Problem: Mixed Methods

Poisson Problem: Saddle-Point Formulation

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We can obtain stable Finite Element discretizations for triangulations \( T_h \). For \( k \geq 1 \), let

Poisson Problem: Stable Mixed Finite Element Spaces
Poisson Problem: Mixed Methods

Poisson Problem: Saddle-Point Formulation

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\[ a(\sigma, \tau) - b(\tau, v) = 0 \]
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We can obtain stable Finite Element discretizations for triangulations \( T_h \). For \( k \geq 1 \), let

Poisson Problem: Stable Mixed Finite Element Spaces

\[ X_h := \left( \mathcal{M}^{k-1} \right)^d = \{ \sigma_h \in L_2(\Omega)^d; \sigma_h|_T \in P_{k-1}, \forall T \in T_h \} \]
We can obtain stable Finite Element discretizations for triangulations $\mathcal{T}_h$. For $k \geq 1$, let

$$X_h := \left( \mathcal{M}^{k-1} \right)^d = \{ \sigma_h \in L_2(\Omega)^d; \sigma_h|_T \in \mathcal{P}_{k-1}, \ \forall T \in \mathcal{T}_h \}$$

$$M_h := \mathcal{M}^k_{0,0} = \{ v_h \in H_0^1(\Omega); \ v_h|_T \in \mathcal{P}_k, \ \forall T \in \mathcal{T}_h \}$$
Poisson Problem: Saddle-Point Formulation

\[ X := L_2(\Omega)^d, \quad M := H_0^1(\Omega), \quad a(\sigma, \tau) := (\sigma, \tau)_{0, \Omega}, \quad b(\tau, v) := -(\tau, \nabla v)_{0, \Omega}. \]

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We can obtain stable Finite Element discretizations for triangulations \( T_h \). For \( k \geq 1 \), let

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\[ M_h := M^k_{0,0} = \{ v_h \in H_0^1(\Omega); v_h|_T \in P_k, \forall T \in T_h \} \]

Note that \( \nabla M_h \subset X_h \), allow us to verify same as in continuous case.
Poisson Problem: Mixed Methods

**Poisson Problem: Saddle-Point Formulation**

\[ X := L_2(\Omega)^d, \quad M := H_0^1(\Omega), \quad a(\sigma, \tau) := (\sigma, \tau)_{0, \Omega}, \quad b(\tau, v) := -\langle \tau, \nabla v \rangle_{0, \Omega}. \]

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**Poisson Problem: Stable Mixed Finite Element Spaces**

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Poisson Problem: Mixed Methods

Raviart-Thomas Element

Let \( h \) denote the mesh size and \( \tau \) be a vector function in \( L^2(\Omega) \). The Raviart-Thomas element \( \tau \) is defined as an element of the space \( RT_k(\mathcal{T}_h) \) such that

\[
\tau \in \{ \tau \in L^2(\Omega) : \tau|_T = a_T + b_T x + c_T y, a_T, b_T, c_T \in P_k, \forall T \in \mathcal{T}_h \}
\]

The\( \tau \cdot n \in \tilde{C}(\partial T) \) denotes that \( \tau \cdot n \) is continuous on the inter-element boundaries. These can be shown to satisfy the Inf-Sup Condition for the Poisson Problem Mixed Formulation.

For \( k = 0 \), \( p \in (P_1)^2 \) has \( p(x, y) = a y + b x + c \) when \( n \) is orthogonal to the line. Edge values determine the polynomial \( p \). Formally, elements are triple \( T, (P_0)^2 + x \cdot P_0, n \cdot p(z_i), i = 1, 2, 3 \) where \( z_i \) is edge midpoint.
Poisson Problem: Mixed Methods

**Raviart-Thomas Element**

\[ X_h := RT_k := \left\{ \tau \in L_2(\Omega)^2; \tau|_T = \begin{pmatrix} a_T \\ b_T \end{pmatrix} + c_T \begin{pmatrix} x \\ y \end{pmatrix}, \; a_T, b_T, c_T \in P_k, \; \forall T \in T_h, \tau \cdot n \in \tilde{C}(\partial T) \right\} \]

\[ M_h := M_k(T_h) := \left\{ v \in L_2(\Omega); v|_T \in P_k, \; \forall T \in T_h \right\} \]

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Edge values determine the polynomial \( p \). Formally, elements are triple \( T, (P_0)^2 + x \cdot P_0, n_i \cdot p(z_i), i = 1, 2, 3 \), \( z_i \) is edge midpoint.
Poisson Problem: Mixed Methods

**Raviart-Thomas Element**

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M_h := \mathcal{M}^k(\mathcal{T}_h) := \{ v \in L_2(\Omega); v|_T \in \mathcal{P}_k, \forall T \in \mathcal{T}_h \}
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Poisson Problem: Mixed Methods

Raviart-Thomas Element

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Raviart-Thomas Element

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Poisson Problem: Mixed Methods

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For $k = 0$, $p \in (P_1)^2$ has

$$p(x, y) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

The $n \cdot p$ is constant on $\alpha x + \beta y = c_0$ when $n$ orthogonal to the line.
The $\tau - n \in \tilde{C}(\partial T)$ denotes that $\tau \cdot n$ is continuous on the inter-element boundaries.

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Raviart-Thomas Element

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The \( n \cdot p \) is constant on \( \alpha x + \beta y = c_0 \) when \( n \) orthogonal to the line. Edge values determine the polynomial \( p \). Formally, elements are triple
Poisson Problem: Mixed Methods

Raviart-Thomas Element

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\[ M_h := \mathcal{M}^k(T_h) := \{ v \in L^2(\Omega); \; v|_T \in P_k, \; \forall T \in \mathcal{T}_h \} \]

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The \( n \cdot p \) is constant on \( \alpha x + \beta y = c_0 \) when \( n \) orthogonal to the line.

Edge values determine the polynomial \( p \). Formally, elements are triple

\[ (T, (P_0)^2 + x \cdot P_0, \; n_i \cdot p(z_i), i = 1, 2, 3, \; z_i \text{ is edge midpoint}) \]

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Finite Element Methods

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Mesh-Dependent Norms:

\[ \| \tau \|_{0,h} := \left( \| \tau \|_0^2 + h \sum_{e \subset \Gamma_h} \| \tau_n \|_0^2 \right)^{1/2} \]

Properties of \( a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega} \):

Ellipticity of \( a(\cdot, \cdot) \) follows from

\[ \| \tau \|_{0,h} \leq C \| \tau \|_{0,\Omega} \quad \forall \tau \in RT_k \Rightarrow a(\tau, \tau) \geq C - h^{-2} \| \tau \|_{0,\Omega}^2 . \]

Properties of \( b(\tau, v) := - (\tau, \nabla v)_{0,\Omega} \):

Use Green's Identity to rewrite as

\[ b(\tau, v) = - \sum_{T \in \mathcal{T}_h} \int_T \tau \cdot \nabla v \, dx + \int_{\Gamma_h} J(v) \tau \, ds . \]

\( J(v) \) is jump of \( v \) in normal direction \( n \). \( \Gamma_h \) := \( S_{\mathcal{T}_h}(\partial T_\Omega) \) interior bnds.

The \( b \) continuity with Mesh-Norms follows readily.

Inf-Sup Condition must be established.
Poisson Problem: Raviart-Thomas Element

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\[ |v|_{1,h} := \left( \sum_{T \in T_h} |v|_{1,T}^2 + h^{-1} \sum_{e \subset \Gamma_h} \| J(v) \|_{0,e}^2 \right)^{1/2} \]
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The \(a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}\) and \(b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}\) defined for \(\tau, \sigma \in L_2(\Omega)^d\).
Mesh-Dependent Norms:

\[ \| \tau \|_{0,h} := \left( \| \tau \|_0^2 + h \sum_{e \subset \Gamma_h} \| \tau n \|_{0,e}^2 \right)^{1/2} \]

\[ | \nu |_{1,h} := \left( \sum_{T \in \mathcal{T}_h} | \nu |_{1,T}^2 + h^{-1} \sum_{e \subset \Gamma_h} \| J(\nu) \|_{0,e}^2 \right)^{1/2} \]

The \( a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega} \) and \( b(\tau, \nu) := - (\tau, \nabla \nu)_{0,\Omega} \) defined for \( \tau, \sigma \in L_2(\Omega)^d \).

Properties of \( a \):

\[ \text{Ellipticity of } a(\cdot, \cdot) \text{ follows from } \| \tau \|_{0,h} \leq C \| \tau \|_{0,\Omega} \Rightarrow a(\tau, \tau) = \| \tau \|_{0,\Omega}^2 \geq C^{-2} \| \tau \|_{0,h}^2. \]
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Mesh-Dependent Norms:

\[
\|\tau\|_{0,h} := \left( \|\tau\|_0^2 + h \sum_{e \subseteq \Gamma_h} \|\tau n\|_{0,e}^2 \right)^{1/2},
\]

\[
|v|_{1,h} := \left( \sum_{T \in \mathcal{T}_h} |v|_{1,T}^2 + \frac{1}{h} \sum_{e \subseteq \Gamma_h} \|J(v)\|_{0,e}^2 \right)^{1/2}.
\]

The \(a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}\) and \(b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}\) defined for \(\tau, \sigma \in L_2(\Omega)^d\).

**Properties of** \(a\): Ellipticity of \(a(\cdot, \cdot)\) follows from

\[
\|\tau\|_{0,h} \leq C \|\tau\|_0, \forall \tau \in RT_k \Rightarrow a(\tau, \tau) = \|\tau\|^2_{0,\Omega} \geq C^{-2} \|\tau\|_{0,h}^2.
\]
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\[ \| \tau \|_{0,h} := \left( \| \tau \|_0^2 + h \sum_{e \subset \Gamma_h} \| \tau n \|_{0,e}^2 \right)^{1/2} \]

\[ |v|_{1,h} := \left( \sum_{T \in T_h} |v|^2_T + h^{-1} \sum_{e \subset \Gamma_h} \| J(v) \|_{0,e}^2 \right)^{1/2}. \]

The \( a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega} \) and \( b(\tau, v) := -(\tau, \nabla v)_{0,\Omega} \) defined for \( \tau, \sigma \in L^2(\Omega)^d \).

**Properties of \( a \):** Ellipticity of \( a(\cdot, \cdot) \) follows from

\[ \| \tau \|_{0,h} \leq C \| \tau \|_0 \quad \forall \tau \in RT_k \Rightarrow a(\tau, \tau) = \| \tau \|_{0,\Omega}^2 \geq C^{-2} \| \tau \|_{0,h}^2. \]

**Properties of \( b \):**

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\]

Properties of \( b \): Use Green’s Identity to rewrite as
Poisson Problem: Raviart-Thomas Element

Mesh-Dependent Norms:

\[
\|\tau\|_{0,h} := \left(\|\tau\|^2_0 + h \sum_{e \subset \Gamma_h} \|\tau n\|^2_{0,e}\right)^{1/2}, \quad |v|_{1,h} := \left(\sum_{T \in T_h} |v|^2_{1,T} + h^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|^2_{0,e}\right)^{1/2}.
\]

The \(a(\sigma, \tau) := (\sigma, \tau)_{0,\Omega}\) and \(b(\tau, v) := -(\tau, \nabla v)_{0,\Omega}\) defined for \(\tau, \sigma \in L^2(\Omega)^d\).

Properties of \(a\): Ellipticity of \(a(\cdot, \cdot)\) follows from

\[
\|\tau\|_{0,h} \leq C \|\tau\|_0, \quad \forall \tau \in RT_k \Rightarrow a(\tau, \tau) = \|\tau\|^2_{0,\Omega} \geq C^{-2} \|\tau\|^2_{0,h}.
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\[
b(\tau, v) = -\sum_{T \in T} \int_T \tau \cdot \nabla v \, dx + \int_{\Gamma_h} J(v) \tau n ds.
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Poisson Problem: Raviart-Thomas Element

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\[ b(\tau, v) = - \sum_{T \in T} \int_T \tau \cdot \text{grad} \ v \, dx + \int_{\Gamma_h} J(v) \tau \, n ds. \]

\( J(v) \) is jump of \( v \) in normal direction \( n \). \( \Gamma_h := \bigcup_T (\partial T \cap \Omega) \) interior bnds.

Raviart-Thomas Element
Poisson Problem: Raviart-Thomas Element

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The \(b\) continuity with Mesh-Norms follows readily.

Inf-Sup Condition must be established.
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

$$\sup_{\tau \in \text{RT}_k} b(\tau, v) \|\tau\|_0, h \geq \beta |v|_1, h, \forall v \in M_k,$$

where $\beta > 0$ and depends on $k$ and the shape regularity of the triangulation $T_h$.

Proof (sketch):
Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$. This implies there exists $\tau \in \text{RT}_0$ so that $\tau_n = h^{-1} J(v)$ on each edge $e \subset \Gamma_h$. Since in this case the area term in Green's Identity for $b$ vanishes, we have

$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 \, ds = h^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_0^2, e = |v|_1^2, h.$$
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Paul J. Atzberger, UCSB

http://atzberger.org/
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

$$\sup_{\tau \in RT_k} \frac{b(\tau, v)}{\|\tau\|_{0,h}} \geq \beta |v|_{1,h}, \quad \forall v \in \mathcal{M}^k,$$
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Finite Element Methods

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Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

$$\sup_{\tau \in RT_k} \frac{b(\tau, \nu)}{\|\tau\|_{0,h}} \geq \beta |\nu|_{1,h}, \quad \forall \nu \in M^k,$$

where $\beta > 0$ and depends on $k$ and the shape regularity of the triangulation $T_h$.

Proof (sketch):
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where $\beta > 0$ and depends on $k$ and the shape regularity of the triangulation $\mathcal{T}_h$.

Proof (sketch):

Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$. 

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Poisson Problem: Raviart-Thomas Element

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Proof (sketch):
Consider case \( k = 0 \), then \( J(v) \) is constant along each edge \( e \subset \Gamma_h \).
This implies there exists \( \tau \in RT_0 \) so that

\[
\tau n = h^{-1} J(v)
\]
on each edge \( e \subset \Gamma_h \).
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

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Proof (sketch):

Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$. This implies there exists $\tau \in RT_0$ so that

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Poisson Problem: Raviart-Thomas Element

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Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$.
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Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$.

This implies there exists $\tau \in RT_0$ so that

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on each edge $e \subset \Gamma_h$. Since in this case the area term in Green’s Identity for $b$ vanishes, we have

$$
b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 ds
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Lemma: The Inf-Sup Condition

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where $\beta > 0$ and depends on $k$ and the shape regularity of the triangulation $T_h$.

Proof (sketch):
Consider case $k = 0$, then $J(\nu)$ is constant along each edge $e \subset \Gamma_h$.
This implies there exists $\tau \in RT_0$ so that

$$\tau n = h^{-1} J(\nu)$$

on each edge $e \subset \Gamma_h$. Since in this case the area term in Green’s Identity for $b$ vanishes, we have

$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(\nu)|^2 ds = c h^{-1} \sum_{e \subset \Gamma_h} \|J(\nu)\|^2_{0,e}$$
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

$$\sup_{\tau \in RT_k} \frac{b(\tau, v)}{\|\tau\|_{0,h}} \geq \beta |v|_{1,h}, \forall v \in M^k,$$

where $\beta > 0$ and depends on $k$ and the shape regularity of the triangulation $T_h$.

**Proof (sketch):**

Consider case $k = 0$, then $J(v)$ is constant along each edge $e \subset \Gamma_h$. This implies there exists $\tau \in RT_0$ so that

$$\tau n = h^{-1} J(v)$$

on each edge $e \subset \Gamma_h$. Since in this case the area term in Green’s Identity for $b$ vanishes, we have

$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 ds = ch^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_{0,e}^2 = |v|_{1,h}^2.$$
Poisson Problem: Raviart-Thomas Element

**Lemma: The Inf-Sup Condition**

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**Proof (sketch) (continued):**

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We also have

$$\|\tau\|_{0,h}^2 \leq ch \sum_{e \subset \Gamma_h} \|\tau\|_{0,e}^2 = ch^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_{0,e}^2 = c|v|_{1,h}^2.$$
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Proof (sketch) (continued):

$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 \, ds = c^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_0^2 = |v|_{1,h}^2.$$ 

We also have

$$\|\tau\|_{0,h}^2 \leq c h \sum_{e \subset \Gamma_h} \|\tau\|_{0,e}^2 = c^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_0^2 = c |v|_{1,h}^2.$$ 

By taking $|v|_{1,h}^2 = |v|_{1,h} c^{-1/2} \|\tau\|_{0,h}$, we have $b(\tau, v) \geq c^{-1/2} |v|_{1,h} \|\tau\|_{0,h}$. 
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

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$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 \, ds = ch^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|^2_{0,e} = |v|_{1,h}^2.$$

We also have

$$\|\tau\|^2_{0,h} \leq ch \sum_{e \subset \Gamma_h} \|\tau\|^2_{0,e} = ch^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|^2_{0,e} = c|v|_{1,h}^2.$$

By taking $|v|_{1,h}^2 = |v|_{1,h} c^{-1/2} \|\tau\|_{0,h}$, we have $b(\tau, v) \geq c^{-1/2} |v|_{1,h} \|\tau\|_{0,h}$. Establishes the Inf-Sup Condition.
Lemma: The Inf-Sup Condition

The bilinear form $b$ with the RT-elements satisfies

$$\sup_{\tau \in RT_k} \frac{b(\tau, v)}{\|\tau\|_{0,h}} \geq \beta |v|_{1,h}, \ \forall v \in M^k,$$

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\[\blacksquare\]
Stokes Hydrodynamic Equations

Stokes Flow

\[
\begin{align*}
\n\Delta u + \nabla p &= -f, \quad x \in \Omega \\
\n\nabla \cdot u &= 0, \quad x \in \Omega \\
\n\n\nabla u &= u_0, \quad x \in \partial \Omega.
\end{align*}
\]

The \( u : \Omega \rightarrow \mathbb{R}^n \) is fluid velocity and \( p : \Omega \rightarrow \mathbb{R} \) is pressure. The \( \nabla \cdot u = 0 \) is constraint for fluid to be incompressible. Only imposes \( p \) up to constant, usually use condition \( \int p dx = 0 \).

Variational Formulation:

\[
X = H^1_0(\Omega)
\]

\[
M = L^2_0(\Omega) := q \in L^2(\Omega); \int q dx = 0.
\]

\[
a(u, v) = \int_\Omega \nabla u : \nabla v \, dx,
\]

\[
b(v, q) = \int_\Omega \nabla \cdot v \, q \, dx.
\]
Stokes Hydrodynamic Equations

Stokes Flow

\[ \Delta u + \text{grad} p = -f, \ x \in \Omega \]
Stokes Hydrodynamic Equations

Stokes Flow

\[ \Delta u + \text{grad } p = -f, \quad x \in \Omega \]

\[ \text{div } u = 0, \quad x \in \Omega \]
Stokes Hydrodynamic Equations

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Stokes Hydrodynamic Equations

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Stokes Flow

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**Variational Formulation:** \( X = H^1_0(\Omega)^n, \ M = L^2_{2,0}(\Omega) := \{ q \in L^2(\Omega); \int q \, dx = 0 \} \).
Stokes Hydrodynamic Equations

**Stokes Flow**

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**Variational Formulation:** \( X = H_0^1(\Omega)^n, \ M = L_{2,0}(\Omega) := \{ q \in L_2(\Omega); \ \int q \, dx = 0 \} \).

\[ a(u,v) = \int_\Omega \text{grad} \ u : \text{grad} \ v \, dx, \]
Stokes Hydrodynamic Equations

Stokes Flow

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Variational Formulation:  \( X = H^1_0(\Omega)^n \), \( M = L^2_{2,0}(\Omega) := \{ q \in L^2(\Omega); \int q dx = 0 \} \).

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**Saddle-Point Problem (Stokes)**

$X = H_0^1(\Omega)^n$, $M = L_{2,0}(\Omega) := \{ q \in L_2(\Omega); \int q dx = 0 \}$. 

Need to establish the Inf-Sup Conditions.
Stokes Hydrodynamic Equations

Variational Formulation: \( X = H^1_0(\Omega)^n,\ M = L_{2,0}(\Omega) := \{ q \in L_2(\Omega); \int q dx = 0 \} \).

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\]

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For Stokes we have

\[ V := \{ v \in X; (\text{div} v, q)_{\Omega} = 0, \forall q \in L^2(\Omega) \}, \]

\[ V_\perp := \{ u \in X; (\nabla u, \nabla v)_{\Omega} = 0, \forall v \in V \}. \]

The \[ V_\perp \] is \[ H^1_0(\Omega) \]-orthogonal complement of \[ V \].

Following two theorems used to establish inf-sup (for proof see literature: Necas 1965, Duvant, Lions 1976).

Theorem I

Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected domain with Lipschitz continuous boundary. The following mappings are isomorphisms

\[ \text{div} : V_\perp \rightarrow L^2(\Omega), v \mapsto \text{div} v. \]

For any \( q \in L^2(\Omega) \) with \( \int_\Omega q \, dx = 0 \), there exists \( v \in V_\perp \subset H^1_0(\Omega) \) with \( \text{div} v = q \) and

\[ \| v \|_{1, \Omega} \leq c \| q \|_{0, \Omega}, \]

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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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\begin{align*}
\text{div} : V^\perp & \rightarrow L^2,0(\Omega) \\
\nu & \mapsto \text{div } \nu.
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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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\[ \mathbf{v} \mapsto \text{div} \, \mathbf{v}. \]

For any \( q \in L_2(\Omega) \) with \( \int q \, dx = 0 \), there exists \( \mathbf{v} \in V^\perp \subset H^1_0(\Omega)^n \).
Stokes Hydrodynamic Equations: Inf-Sup Conditions

For Stokes we have

\[ V := \{ v \in X; (\text{div}\, v, q)_{0,\Omega} = 0, \forall q \in L_2(\Omega) \}, \quad V^\bot := \{ u \in X; (\text{grad}\, u, \text{grad}\, v)_{0,\Omega} = 0, \forall v \in V \}. \]

The \( V^\bot \) is \( H^1 \)-orthogonal complement of \( V \).

Following two theorems used to establish inf-sup (for proof see literature: Necas 1965, Duvant, Lions 1976).

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v \mapsto \text{div} v.
\]

For any \( q \in L_2(\Omega) \) with \( \int q \, dx = 0 \), there exists \( v \in V^\bot \subset H_0^1(\Omega)^n \) with

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\text{div} v = q \text{ and } \|v\|_{1,\Omega} \leq c\|q\|_{0,\Omega},
\]
For Stokes we have

\[ V := \{ \mathbf{v} \in X; (\text{div} \mathbf{v}, q)_{0,\Omega} = 0, \forall q \in L_2(\Omega) \}, \quad V^\perp := \{ u \in X; (\text{grad} u, \text{grad} \mathbf{v})_{0,\Omega} = 0, \forall \mathbf{v} \in V \}. \]

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\text{div} \mathbf{v} = q \quad \text{and} \quad ||v||_{1,\Omega} \leq c ||q||_{0,\Omega},
\]

where \( c = c(\Omega) \) constant.
Stokes Hydrodynamic Equations: Inf-Sup Conditions

Theorem II

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with Lipschitz continuous boundary.

1. For the following linear mapping, the image is closed in $H^{-1}(\Omega)$: $\text{grad} : L^2(\Omega) \rightarrow H^{-1}(\Omega)$.

2. For $f \in H^{-1}(\Omega)$, if $\langle f, v \rangle = 0$, $\forall v \in V$.

3. There is constant $c = c(\Omega)$ so that $\|q\|_{0,\Omega} \leq c (\|\text{grad} q\|_{-1,\Omega} + \|q\|_{-1,\Omega})$, $\forall q \in L^2(\Omega)$.

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   \[
   \| q \|_{0, \Omega} \leq c \left( \| \text{grad} q \|^{-1}_{-1, \Omega} + \| q \|^{-1}_{-1, \Omega} \right) \forall q \in L^2(\Omega),
   \]
   \[
   \| q \|_{0, \Omega} \leq c \| \text{grad} q \|^{-1}_{-1, \Omega} \forall q \in L^2(\Omega).
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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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   \[
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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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   \]

   \[
   \|q\|_{0,\Omega} \leq c \|\text{grad } q\|_{-1,\Omega} \quad \forall q \in L^2,0(\Omega).
   \]
Lemma: Inf-Sup for Stokes

\[ \sup_{v \in X} (v, q) \|v\|_1 \geq \beta \|q\|_0. \]

Proof (sketch):

(By Theorem I):

For a \( q \in L^2_0 \), exists \( v \in H^1_0(\Omega) \) satisfying \( \text{div} v = q \) and \( \|v\|_{1,\Omega} \leq c \|q\|_{0,\Omega} \) (from previous thm.).

This implies

\[ \sup_{v \in X} (v, q) \|v\|_1 = (\text{div} v, q) \|v\|_1 = \|q\|_{2,0} \|v\|_1 \geq \|q\|_{2,0} c \|q\|_{0,\Omega} = 1. \]

This gives the Brezzi Condition for \( b \).
Lemma: Inf-Sup for Stokes

\[ \sup_{v \in X} \frac{b(v, q)}{\|v\|_1} \geq \beta \|q\|_0. \]
 Lemma: Inf-Sup for Stokes

$$\sup_{v \in X} \frac{b(v, q)}{\|v\|_1} \geq \beta \|q\|_0.$$
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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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**Lemma: Inf-Sup for Stokes**

\[ \sup_{\nu \in X} \frac{b(\nu, q)}{\|\nu\|_1} \geq \beta \|q\|_0. \]

**Proof (sketch):**

*(By Theorem I):* For a \( q \in L^2_{2,0} \), exists \( \nu \in H^1_0(\Omega)^n \) satisfying \( \text{div} \, \nu = q \) and \( \|\nu\|_{1,\Omega} \leq c \|q\|_{0,\Omega} \) (from previous thm.) This implies

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Stokes Hydrodynamic Equations: Inf-Sup Conditions

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This gives the Brezzi Condition for \( b \).
Lemma: Inf-Sup for Stokes

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\]

Proof (sketch):

(By Theorem II): For \( q \in L^2(\Omega) \), 0, and second inequality of II, that

\[
\|\nabla q\|_{-1} \geq c_{-1} \|q\|_0.
\]

From def. of negative norm, there exists \( v \in H^1_0(\Omega) \) with \( \|v\|_1 = 1 \) and

\[
(b(v, q), \Omega) \geq \frac{1}{2} \beta \|v\|_1 \|\nabla q\|_{-1} \geq \frac{1}{2} \beta \|q\|_0.
\]

By Greens Identity

\[
(b(-v, q), \Omega) = -\int v \cdot \nabla q \, dx
\]

we have

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\[ \|\nabla q\|^{-1} \geq c^{-1} \|q\|_0. \]

From definition of negative norm, there exists \(v \in H^1_0(\Omega)\) with 
\[ \|v\|_1 = 1 \text{ and } (v, \nabla q)_{\Omega} \geq \frac{1}{2} \|v\|_1 \|\nabla q\|^{-1} \geq \frac{1}{2} c \|q\|_0. \]

By Green's Identity 
\[ \langle v, q \rangle = -\int_{\Omega} v \cdot \nabla q \, dx \]

we have 
\[ \langle -v, q \rangle \geq \frac{1}{2} c \|q\|_0. \]

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\[ \sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \geq \beta \|q\|_0. \]

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(By Theorem II):

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\sup_{v \in X} \frac{b(v, q)}{\|v\|_1} \geq \beta \|q\|_0.
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Proof (sketch):

(By Theorem II): For \( q \in L^2,0 \), and second inequality of II, that \( \|\text{grad} \ q\|_1 \geq c^{-1} \|q\|_0 \). From def. of negative norm, there exists \( v \in H^1_0(\Omega)^n \) with \( \|v\|_1 = 1 \) and
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This gives the Brezzi Condition for \( b \).
Stokes Hydrodynamic Equations: Inf-Sup Conditions

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\[\square\]
Consider triangulation $\mathcal{T}_h$ and polynomial shape spaces $\mathcal{P}_j$. 

\[ X_h := M_{2,0} \begin{pmatrix} n_v \in C(\bar{\Omega}) \setminus H^1_0(\Omega) \\ v_h |_T \in P_2, \forall T \in \mathcal{T}_h \end{pmatrix} \]

\[ M_h := M_{1,0} \begin{pmatrix} n_q \in C(\Omega) \setminus L^2_0(\Omega) \\ q_h |_T \in P_1, T \in \mathcal{T}_h \end{pmatrix} \]
Stokes Hydrodynamic Equations: Taylor-Hood Element

Consider triangulation $\mathcal{T}_h$ and polynomial shape spaces $\mathcal{P}_j$.

**Taylor-Hood Elements:** Stability achieved by velocity field in polynomial space larger degree than the pressure space.
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**Modified Taylor-Hood Element:** Use piece-wise linear functions on sub-triangles (macro element)

- Figure: $\times$ denotes pressure values, $\cdot$ denotes velocity values.
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**Figure:** $\times$ denotes pressure values, $\cdot$ denotes velocity values.
MINI Elements: Achieves stability by using interior "bubble" elements.

For triangle, let \( \lambda_1, \lambda_2, \lambda_3 \) denote the barycentric coordinates of a point \( x \).

Add to the shape space the "bubble" function \( b(x) = \lambda_1 \lambda_2 \lambda_3 \).

Note, \( b \) vanishes on boundary of \( T \).

The finite element spaces are \( X_h = H^1_0(\Omega) \oplus B_3 \), \( M_h = M_0(\Omega) \), where \( B_3 = \{ v \in C^0(\bar{\Omega}); v \mid_T \in \text{span}\{\lambda_1 \lambda_2 \lambda_3\}, \forall T \in \mathcal{T}_h \} \).

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