Sobolev Spaces

Paul J. Atzberger

206D: Finite Element Methods
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Basic Definitions

The $L^2(\Omega)$ for a smooth domain $\Omega$, denotes the space of all functions $f$ that are Lebesgue square-integrable $\int_{\Omega} f^2 \, dx < \infty$. The $L^2$ norm is defined as:

$$\|u\|_2 = \left( \int_{\Omega} u^2 \, dx \right)^{1/2}$$

Definition: A function $u \in L^2(\Omega)$ has as its weak derivative $v = D^\alpha u = \partial^\alpha u$ if

$$(v, w)_{L^2(\Omega)} = (-1)^{|\alpha|} (u, \partial^\alpha w)_{L^2(\Omega)}, \quad \forall w \in C^\infty_0(\Omega)$$

$C^\infty_0(\Omega)$ is the space of all functions that are infinitely continuously differentiable and vanish outside a compact set.
Basic Definitions

The $L^2(Ω)$ for a smooth domain $Ω$, denotes the space of all functions $f$ that are Lebegue square-integrable $\int_{Ω} f^2 \, dx < \infty$. We define the $L^2$-inner-product as

$$(u,v)_0 = (u,v)_{L^2} = \int_{Ω} u(x)v(x) \, dx.$$
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A function $u \in L^2$ has as its **weak derivative** $v = D_\alpha u = \partial^\alpha u$ if

$$(v, w)_{L^2} = (-1)^{|\alpha|} (u, \partial^\alpha w)_{L^2}, \ \forall w \in C_0^\infty.$$  

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$C^\infty$ is the space of all functions is infinitely continuously differentiable. The $C_0^\infty \subset C^\infty$ are all functions zero outside a compact set.
For any integer $m \geq 0$, let $H^m$ be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order $m$, $|\alpha| \leq m$. 

We define an inner-product on $H^m$ as 

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v).$$ 

We define $H^m$-norm as 

$$\|u\|_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}}.$$ 

We define $k$-semi-norm as 

$$|u|_k = \sqrt{\sum_{|\alpha| = k} (\partial^\alpha u, \partial^\alpha u)_0} = \sqrt{\sum_{|\alpha| = k} \|\partial^\alpha u\|_{L^2}}.$$ 

We refer to $H^m$ with this inner-product as a Sobolev space. Also denoted by $W^{m,2}$. 

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We refer to $H^m$ with this inner-product as a **Sobolev space**. Also denoted by $W^{m, 2}$. 
Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem
Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\| \cdot \|_m$.

This means that we can view $H^m$ as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $\langle \cdot , \cdot \rangle_m$.

The $H^m$ is the completion under $\| \cdot \|_m$.

Definition
Denote the completion of $C^\infty_0(\Omega)$ under $\| \cdot \|_m$ by $H^m_0(\Omega)$.

We have the following relations between the function spaces $L^2(\Omega) = H^0(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \cdots \supset H^m(\Omega) \cup \cdots$.
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**Definition**

Denote the completion of \( C_0^\infty(\Omega) \) under \( \| \cdot \|_m \) by \( H^m_0(\Omega) \).

We have the following relations between the function spaces

\[
L^2(\Omega) = H^0(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \ldots \supset H^m(\Omega) \quad \text{and} \quad H^0_0(\Omega) \supset H^1_0(\Omega) \supset H^2_0(\Omega) \ldots \supset H^m_0(\Omega).
\]
We can also define function spaces based on $L^p(\Omega)$ and $C_0$ similarly using the norm $\| \cdot \|_p$. 

**Definition** 

The Sobolev space denoted by $W^{m,p}$ (also by $W^{m,p}_0$) is the collection of functions obtained by completing $C_\infty(\Omega) \subset L^p(\Omega)$ under the norm $\| \cdot \|_m$. Similarly, we obtain $W^{m,p}_0$ by completing $C_\infty_0(\Omega) \subset L^p(\Omega)$ under $\| \cdot \|_m$. 

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Similarly, we obtain $W^{m,p}_0$ by completing $C^\infty_0(\Omega) \subset L^p(\Omega)$ under $\| \cdot \|_m$. 
Definition

Consider a given domain $\Omega$ and compact sets $K \subset \Omega$. We define the set of \textbf{locally integrable} functions as

$$L^1_{\text{loc}}(\Omega) := \{ v | v \in L^1(K), \forall K \subset \Omega^o \}$$
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These functions can behave poorly near the boundary of $\Omega$ as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial \Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.
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The $p = \infty$ norm is given by

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If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).
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**Example:** Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$. 

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Sobolev Spaces

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Consider a given domain \( \Omega \) and compact sets \( K \subset \Omega \). We define the set of *locally integrable* functions as

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L^1_{\text{loc}}(\Omega) := \{ v | v \in L^1(K), \forall K \subset \Omega^o \}
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These functions can behave poorly near the boundary of \( \Omega \) as illustrated by \( v(x) = \phi(1/\text{dist}(x, \partial \Omega)) \) where \( \phi(x) = e^{e^x} \) which still yields \( v \in L^1_{\text{loc}}(\Omega) \).

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The \( p = \infty \) norm is given by

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\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| | x \in \Omega\}
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Sobolev Spaces

Definition

For $1 \leq p < \infty$, we define the Sobolev norm as

$$\|v\|_{W^k_p(\Omega)} := \left( \sum_{|\alpha| \leq k} \|D^\alpha_w v\|_{L^p(\Omega)}^p \right)^{1/p},$$

where $k$ is a non-negative integer, $v \in L^1_{\text{loc}}(\Omega)$, and $D^\alpha_w v$ exists for all $|\alpha| \leq k$. For $p = \infty$, we define the Sobolev norm as

$$\|v\|_{W^k_\infty(\Omega)} := \max_{|\alpha| \leq k} \|D^\alpha_w v\|_{L^\infty(\Omega)}.$$
For $1 \leq p < \infty$, we define the **Sobolev norm** as

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\|v\|_{W_p^k(\Omega)} := \left( \sum_{|\alpha| \leq k} \|D^\alpha_w v\|_{L_p(\Omega)}^p \right)^{1/p},
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We assume $k$ is a non-negative integer, $v \in L_{\text{loc}}^1(\Omega)$, and $D^\alpha_w v$ exists for all $|\alpha| \leq k$. 
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For \( 1 \leq p < \infty \), we define the Sobolev semi-norm as

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The general Sobolev spaces also satisfy inclusion relations.
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**Theorem**
For $k, m$ are non-negative integers with $k \leq m$ and $p$ any real number with $1 \leq p \leq \infty$, we have

$$W_p^m(\Omega) \subset W_p^k(\Omega).$$
The general Sobolev spaces also satisfy inclusion relations.

**Theorem**

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**Theorem**

For $k$ any non-negative integer and $p, q$ any real numbers with $1 \leq p \leq q \leq \infty$, we have

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**Theorem**

For $k, m$ non-negative integers with $k < m$ and and $p, q$ any real numbers with $1 \leq p < q \leq \infty$, we have

$$W^m_q(\Omega) \subset W^k_p(\Omega).$$
Poincaré-Friedrichs Inequality:

Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H^1_0(\Omega).$$
Poincaré-Friedrichs Inequality

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This shows the 1-semi-norm bounds the 0-norm.
Poincaré-Friedrichs Inequality

**Theorem**

**Poincaré-Friedrichs Inequality:** Consider domain \( \Omega \subset Q = [0, s]^n \), \( Q \) is cube of side-length \( s \). Then

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\|v\|_0 \leq s|v|_1, \ \forall v \in H^1_0(\Omega).
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**Proof:**
Poincaré-Friedrichs Inequality

**Theorem**

**Poincaré-Friedrichs Inequality:** Consider domain $\Omega \subset Q = [0, s]^n$, $Q$ is cube of side-length $s$. Then

$$\|v\|_0 \leq s|v|_1, \forall v \in H^1_0(\Omega).$$

**Proof:** Since $v \in H^1_0$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express $v$ as

$$v(x_1, x_2, \ldots, x_n) = v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial_1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial_1 v(z, x_2, \ldots, x_n) dz$$
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v(x_1, x_2, \ldots, x_n) = v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz
\]

By the Cauchy-Swartz inequality we have

\[
|v(x)|^2 \leq \left( \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 = \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
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v(x_1, x_2, \ldots, x_n) = v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz
\]

By the Cauchy-Swartz inequality we have

\[
|v(x)|^2 \leq \left( \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 = \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
\]

\[
\leq s \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
\]
Poincaré-Friedrichs Inequality

Theorem

**Poincaré-Friedrichs Inequality:** Consider domain \( \Omega \subset Q = [0, s]^n \), \( Q \) is cube of side-length \( s \). Then

\[
\|v\|_0 \leq s |v|_1, \; \forall v \in H^1_0(\Omega).
\]

**Proof:** Since \( v \in H^1_0 \) and using a point on the boundary \((0, x_2, x_3, \ldots, x_n)\) we can express \( v \) as

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\]

\[
\leq s \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 \, dz
\]

We integrate over the cube \( Q = [0, s]^n \) with \( v, \partial^1 v \) extended to vanish outside of \( \Omega \).
Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

$$\|v\|_0 \leq s |v|_1, \ \forall v \in H^1_0(\Omega).$$

Proof:

$$|v(x)|^2 \leq \left( \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 = \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \leq s \int_0^{s} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$
**Poincaré-Friedrichs Inequality**

Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

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$$\|v\|_0 = \int_Q |v(x)|^2 dx \leq s^2 \int_Q |\partial_1 v(x)|^2 dx = |v|_1.$$
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Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives \( \tilde{v} = \partial^{\alpha} u \) to obtain

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**Theorem**

**Poincaré-Friedrichs Inequality II:** Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length $s$. Then

$$|v|_m \leq \|v\|_m \leq (1 + s)^m |v|_m, \ \forall v \in H^m_0(\Omega).$$
We can also apply the inequality using the derivatives \( \tilde{v} = \partial^{\alpha} u \) to obtain

\[
|\partial^{\alpha} u|_0 \leq s |\partial^{1\alpha} u|_0, \quad |\alpha| \leq m - 1, \quad u \in H_0^m(\Omega).
\]

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**Theorem**

**Poincaré-Friedrichs Inequality II:** Consider the domain \( \Omega \subset [0, s]^n \) is contained within a cube of side-length \( s \). Then

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When \( \Omega \) is bounded, the \( m \)-semi-norm \( |v|_m \) is in fact a proper norm on \( H_0^m(\Omega) \).
We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

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When $\Omega$ is bounded, the $m$-semi-norm $|v|_m$ is in fact a proper norm on $H^m_0(\Omega)$.

The norm $|v|_m$ is equivalent to $\|v\|_m$ (convergence in one implies convergence in other).
Sobolev Inequality

Theorem

**Sobolev Inequality:** Consider a domain $\Omega$ with Lipschitz boundary, $k > 0$ with $k$ an integer, and $p$ real number with $1 \leq p < \infty$ such that

We then have there is a constant $C$ so that for all $u \in W^{k,p}(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}.$$

Also, for the equivalence class of $u$ in $L^\infty(\Omega)$, there is a representative that is a continuous function.

Significance: Shows that if a function has enough weak derivatives then in fact it can be viewed as equivalent to a continuous, bounded function. Also, shows that if we have convergence in $\|\cdot\|_{W^{k,p}(\Omega)}$ then also converges in $\|\cdot\|_{L^\infty(\Omega)}$. 

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Finite Element Methods

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Trace Theorems (boundary conditions)

When working with $L^p$ functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x, y) | x^2 + y^2 < 1\} = \{(r, \theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma: Let $\Omega$ be the unit disk. For all $u \in W^{1,2}(\Omega)$ the restriction of $u |_{\partial \Omega}$ can be interpreted as a function in $L^2(\partial \Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial \Omega)} \leq \frac{8}{14} \|u\|_{W^{1,2}(\Omega)}.$$

Proof (sketch): For $u \in C^1(\Omega)$, consider the restriction to $\partial \Omega$ when $r \leq 1$.

$$u(1, \theta)^2 = \int_0^1 \partial_r (r^2 u(r, \theta)^2) dr = \int_0^1 2 (r^2 u u_r + ru u_r^2) (r, \theta) dr \leq \int_0^1 2 (r^2 |u||\nabla u| + ru u_r^2) (r, \theta) dr.$$
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Let $\Omega$ be the unit disk. For all $u \in W^1_2(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W^1_2(\Omega)}^{1/2}.$$

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Lemma

Let \( \Omega \) be the unit disk. For all \( u \in W^1_2(\Omega) \) the restriction of \( u|_{\partial \Omega} \) can interpreted as a function in \( L^2(\partial \Omega) \). Furthermore, it satisfies the bound

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\|u\|_{L^2(\partial \Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W^1_2(\Omega)}^{1/2}.
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Proof (sketch):

\[
u(1, \theta)^2 \leq \int_0^1 2 \left(r^2 |u||\nabla u| + ru^2\right) (r, \theta) dr \leq \int_0^1 2 \left(|u||\nabla u| + u^2\right) (r, \theta) dr.
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Using polar coordinates and integrating out the \( \theta \) we obtain
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Proof (sketch):

$$u(1, \theta)^2 \leq \int_0^1 2 (r^2|u||\nabla u| + ru^2)) (r, \theta)dr \leq \int_0^1 2 (|u||\nabla u| + u^2)) (r, \theta)dr.$$ 

Using polar coordinates and integrating out the $\theta$ we obtain

$$\int_{\partial \Omega} u^2 d\theta \leq 2 \int_{\Omega} (|u||\nabla u| + u^2) dxdy.$$
Lemma

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The norm of function $u|_{\partial \Omega}$ restricted to the boundary is

$$\|u\|_{L^2(\partial \Omega)}^2 := \int_{\partial \Omega} u^2 d\theta = \int_0^{2\pi} u(1, \theta)^2 d\theta.$$
Lemma

Let $\Omega$ be the unit disk. For all $u \in W^1_2(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W^1_2(\Omega)}^{1/2}.$$ 

Proof (sketch):

By Cauchy-Swartz we have
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Proof (sketch):

By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial \Omega)}^2 \leq 2 \|u\|_{L^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dxdy \right)^{1/2} + 2 \int_{\Omega} u^2 \, dxdy.$$
Lemma

Let Ω be the unit disk. For all \( u \in W^1_2(\Omega) \) the restriction of \( u|_{\partial\Omega} \) can interpreted as a function in \( L^2(\partial\Omega) \). Furthermore, it satisfies the bound

\[
\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|^{1/2}_{L^2(\Omega)} \|u\|^{1/2}_{W^1_2(\Omega)}.
\]

Proof (sketch):

By Cauchy-Swartz we have

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\|u\|_{L^2(\partial\Omega)}^2 \leq 2 \|u\|_{L^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + 2 \int_{\Omega} u^2 \, dx \, dy.
\]

Using the arithmetic-geometric mean inequality we have
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Let $\Omega$ be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial \Omega}$ can interpreted as a function in $L^2(\partial \Omega)$. Furthermore, it satisfies the bound

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$$\|u\|_{L^2(\partial \Omega)}^2 \leq 2\|u\|_{L^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dx \, dy \right)^{1/2} + 2 \int_{\Omega} u^2 \, dx \, dy.$$  

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Lemma

Let $\Omega$ be the unit disk. For all $u \in W_{2}^{1}(\Omega)$ the restriction of $u|_{\partial \Omega}$ can be interpreted as a function in $L^{2}(\partial \Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^{2}(\partial \Omega)} \leq 8^{1/4} \|u\|_{L^{2}(\Omega)}^{1/2} \|u\|_{W_{2}^{1}(\Omega)}^{1/2}.$$ 

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Paul J. Atzberger, UCSB

Finite Element Methods

http://atzberger.org/
Trace Theorems (boundary conditions)

**Theorem**

**Trace Theorem:** Consider \( \Omega \) with a Lipschitz boundary and \( p \) real number with \( 1 \leq p \leq \infty \). We then have there exists a constant \( C \) so that

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\|v\|_{L^p(\partial \Omega)} \leq C \|v\|_{L^p(\Omega)}^{1-1/p} \|v\|_{W^1_p(\Omega)}^{1/p}, \quad \forall v \in W^1_p(\Omega).
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