Final Projects
Introduction to Numerical Analysis
Professor: Paul J. Atzberger

Due Date: Friday, August 7th

Instructions: In the final project you are to apply the numerical methods developed in the class to one of the following problems from physics/engineering or finance. For your assigned project you are to write a report which answers each of the stated questions. You are to give your numerical results in the form of a well-organized table or list as part of the overall discussion of the report. All codes should be written from scratch in Matlab/Octave making use of the algorithms discussed in class.

Turn into my mailbox in South Hall 6th floor by 5:00pm.
Project 1: (Physics and Engineering) Suppose that you are faced with the task of planning a highway between two important cities separated by a mountainous landscape. Building the highway is expected to be very costly given the expansive distances involved and the rugged terrain. Different choices for the path followed in the terrain by the highway could have major economic consequences. In this project you will consider a model for the costs of these different choices and determine a good plan for the highway.

The path taken by the highway will be represented by the parameterized curve given by \( x = \gamma_1(s) \) and \( y = \gamma_2(s) \), which we shall denote by the vector valued function \( \gamma(s) = (\gamma_1(s), \gamma_2(s)) \). To measure the relative costs for different plans for the highway following different paths we shall use the utility function:

\[
E[\gamma] = \int_0^L f(\gamma_1(s), \gamma_2(s)) ds.
\]

The function \( f(x, y) \) gives the local cost required if the highway goes through the point \((x, y)\). This can be thought of as the local cost of having to remove boulders, drain swampland, or dynamite a hill. In the notation, \( L \) is the arc length of the curve \( \gamma \).

Finding the optimal highway plan requires minimizing over arbitrary paths \( \gamma(s) \), which in general is a challenging problem. In this project you will explore one approach to constructing approximate solutions to this problem.

(a) Write a function in Matlab/Octave called \texttt{arcLengthPath} which for a given collection of ordered points \((x_0, y_0), \cdots, (x_N, y_N)\) gives the approximate arc length. Compute this estimate by using a piecewise linear interpolation of the points and summing the length of each of the linear segments.

(b) Write a function in Matlab/Octave called \texttt{interpPath} which interpolates a given collection of ordered points \((x_0, y_0), \cdots, (x_N, y_N)\) using a natural cubic spline. For the given arc length \( s \) use the method to compute a point \((x(s), y(s))\) having approximate arc length \( s \) along the curve. (Hint: For each point \((x_j, y_j)\) use the arc length \( s_j \) from part (a) and interpolate separately the data for \( x(s) \) given by \( \{(s_j, x_j)\} \) and the data for \( y(s) \) given by \( \{(s_j, y_j)\} \).)

(c) Write a function in Matlab/Octave called \texttt{updatePoint} which takes as input a parameter \( h \) and for a given point \((x, y)\) computes \( \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \) and gives the new point \((x - h \frac{\partial f}{\partial x}, y - h \frac{\partial f}{\partial y})\) as output. Note this moves each of the points in the direction of steepest descent on the surface defined by \((x, y, f(x, y))\).

(d) In order to find a descent route for the highway connecting \((x_0, y_0)\) and \((x_N, y_N)\) first consider an initial path obtained by linear interpolation \((x_j, y_j) = (1 - \alpha_j)(x_0, y_0) + \alpha_j(x_N, y_N)\), where \( \alpha_j = j/N \). We shall iteratively attempt to find successively better routes by using the following strategy:

(i) Move each of the points \((x_j, y_j)\) with indices \( j = 1, \cdots, N - 1 \) in the direction of steepest descent on the landscape using the procedure from step (c) above.
(ii) Redistribute the points on the path to be the same arc length distance apart with neighbors to help ensure our discrete points give a good representation of the path and to avoid clustering.

(iii) Repeat step (i) until path appears to have converged.

The first step (i) can be accomplished by calling `updatePoint` on each of the points $(x_j, y_j)$ in the range $j = 1, \ldots, N - 1$. The second step (ii) can be accomplished by calling `arcLengthPath` and then calling `interpPath` for each arc length $s_j = jL/N$ with $j = 1, \ldots, N - 1$ to obtain the new $(x_j, y_j)$. While this iteration scheme will not strictly give the optimal paths of $E$, it is expected to give decent results for the path of the highway.

Write a Matlab/Octave code to carry out this strategy. Use the code for the initial path given by linear interpolation between $(x_0, y_0) = (0.5, 0)$ and $(x_N, y_N) = (0.5, 1)$ with $N$ at least 10 when the landscape is given by $f(x, y) = 1 - r(r-1)^2$, where $r = \sqrt{(x - 0.52)^2 + (y - 0.51)^2}$. Run your code until successive iterates appear to change very little between iterations, say by no more than $10^{-2}$. What values for the cost $E$ does the initial route and the one you found give? By what factor has $E$ improved? Give a plot of the landscape $f(x, y)$ and the route $\gamma(s)$ you obtained. (Hint: Use Matlab/Octave `surf()`, `mesh()`, `plot()`, `holdon`, commands).

(e) Using your code from part (d) find the optimal route from $(x_0, y_0) = (0, 0)$ to $(x_N, y_N) = (1, 1)$ for the the landscape $f(x, y) = 3(\cos^2(2\pi k x) \sin^2(2\pi k y) + 1)$, $(k = 2)$. Give a plot of the landscape $f(x, y)$ and the route $\gamma(s)$ you obtain.
Project 2: (Black-Scholes-Merton Option Pricing) When making investments in an asset in the marketplace (such as a stock) there are typically substantial risks in the future value of the asset. To facilitate management of these risks, banks sell contracts to protect investors against either large increases or large decreases in the value of an asset. A common class of contracts used for this purpose are referred to as options. For example, to manage price changes of a stock a European Put Option is a contract which gives the holder the right to sell at a specified price $K$ (called the strike price) a given number of units of the stock at a specific time in the future $T$ (called the maturity time). Similarly, a European Call Option gives the holder the right to buy at a specified price $K$ at a specific time $T$ in the future.

An important problem for banks who buy and sell such contracts is to determine a reasonable price for the contracts. Given the nature of the contracts, the price charged by a bank must somehow reflect the current value of the asset in the marketplace while at the same time reflecting the future liabilities the bank assumes by issuing the contract. This is in general a challenging problem given all of the uncertainties in the future behavior of the marketplace. However, when certain assumptions are made about the marketplace such a price can be determined. A well-known approach used in practice is the Black-Scholes-Merton Option Pricing Theory, see Options, Futures, and Other Derivatives by J. Hull, if you are interested in more details.

From the Black-Scholes-Merton Option Pricing Theory the price of a European call option $(c)$ and put option $(p)$ are given by:

$$
c(s_0, K, T) = s_0 N(d_1) - Ke^{-rT} N(d_2)
$$

$$
p(s_0, K, T) = Ke^{-rT} N(-d_2) - s_0 N(-d_1)
$$

where $s_0$ is the current price of the stock (spot price), $K$ is the agreed upon selling price (strike price), and $T$ is the time at which the contract may be executed (maturity time). We also have

$$
d_1 = \frac{1}{\sigma \sqrt{T}} \left( \log \left( \frac{s_0}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T \right)
$$

$$
d_2 = \frac{1}{\sigma \sqrt{T}} \left( \log \left( \frac{s_0}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T \right)
$$

where $r$ is the current compounding interest rate and $\sigma^2$ is a parameter characterizing how much prices are expected to fluctuate in the marketplace (volatility). In the notation $N$ denotes the cumulative normal distribution function:

$$
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy.
$$

We also remark that $d_2 = d_1 - \sigma \sqrt{T}$.

In order to use this formula in practice the expressions above must be numerically evaluated. In this project you will develop Matlab/Octave codes for the algorithms discussed in class to price various call and put options.
(a) Implement a Matlab/Octave code for a function called \( \text{cumulativeNormal} \) which takes \( d \) as input. The function should use the methods discussed in class for approximating integrals. To approximate the \( y_1 = -\infty \) term use at least \( y_1 \leq -10 \) in the range of the integral. What value of the grid spacing \( h \) ensures the integral is evaluated to 3 significant digits when \( d = 0 \)? Note that for \( d = 0 \) the exact value is \( 1/2 \).

(b) Suppose the current price of Microsoft stock is $110 today, compute the price of a put option which gives a holder the right to sell the stock 100 days from today for $109. Assume that the current compounding interest rate \( r = 0.05/365 \) and the market volatility is \( \sigma^2 = 0.2/365 \). Note that this means \( s_0 = 110, K = 109, T = 100 \).

(c) Make a plot of the the price of the put option in part (b) as the strike price \( K \) is varied from 80 to 160. Give a labeled plot of \( p \) vs. \( K \). Discuss what happens to the price of the put option as the strike price is decreased? Can you explain intuitively why this is expected to happen?

(d) Suppose the current price of Apple stock is $110 today, compute the price of a call option which gives a holder the right to buy the stock 100 days from today for $109. Assume that the current compounding interest rate \( r = 0.05/365 \) and the market volatility is \( \sigma^2 = 0.2/365 \). (Note: \( s_0 = 110, K = 109, T = 100 \))

(e) Make a plot of the the price of the call option in part (b) as the strike price \( K \) is varied from 80 to 160. Give a labeled plot of \( c \) vs \( K \). Discuss what happens to the price of the call option as the strike price is decreased? Can you explain intuitively why this happens?

(f) Suppose an investor goes long (buys) a call option and goes short (sells) a put option. Using your data from part (c) and (e) make a plot of \( c - p \) vs \( K \) (now assuming they are for the same underlying stock). The price of a forward contract, in which two parties agree to a selling price of an asset at some future time \( T \) is given by \( f(s_0, K, T) = s_0 - e^{-rT}K \). Add to your plot of the portfolio \( c - p \) a plot of the price of the forward contract. How do they compare? Can you explain why? The strike price at which the forward contract is worth 0 is call the ”fair price” of the contract. At this strike price neither party makes a net financial gain by entering into the contract. For what value of \( K \) does the forward become worth zero for the contract with the parameters above? This relationship between the forward contract and the portfolio of the European call and put options is called ”put-call parity”.

Project 3: (Markowitz Portfolio Theory) When making investments in the marketplace a trade-off usually needs to be made between the expected return (payoff) of an asset and the riskiness in obtaining that return or a loss. The central tenet of Markowitz Portfolio Theory is that if two investment opportunities have the same expected return, then the one which is less risky is more desirable to investors. When faced with the opportunity to invest in many different assets an interesting problem arises in how to choose an optimal portfolio, which attempts to maximize the investment return while minimizing risks. In this project you will explore one model of investments which attempts to capture these trade-offs and use this model to construct optimal portfolios for a collection of assets.

In the notation we shall denote the expected return of the $i^{th}$ asset by $\mu_i$ and the riskiness of the asset by $\sigma_i^2$, the variance.

(a) Consider two assets with expected returns $\mu_1$, $\mu_2$, variance $\sigma_1^2$, $\sigma_2^2$, and covariance $\sigma_{1,2}$. From Markowitz Portfolio Theory a portfolio which invests a fraction $w_1$ of an investor’s wealth in asset 1 and a fraction $w_2$ in asset 2 has the expected return:

$$\mu_p = w_1 \mu_1 + w_2 \mu_2$$ (1)

and variance

$$\sigma_p^2 = w_1^2 \sigma_1^2 + 2w_1w_2\sigma_{1,2} + w_2^2 \sigma_2^2.$$ (2)

Now let us suppose an investor wishes to invest $w = $100,000 in the assets to attain a return $\mu_p = 0.08$. We can express the weights as $w_1 = (1 - \alpha)$, $w_2 = \alpha$. Let the assets have $\mu_1 = 0.03$, $\mu_2 = 0.09$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\sigma_{1,2} = -0.02$. Write a Matlab/Octave code which implements both a Bisection Method and a Newton Method to find the value of $\alpha$ which makes $\mu_p = 0.08$ in equation 1. This is easily solved analytically, so check your code returns the correct result. What are the weights $w_1$, $w_2$? What is the variance $\sigma_p^2$ of the portfolio with return 0.08? Using that the fluctuations in the future value of the investment is modeled by the range $V_1(T) = \mu_p T - \sigma_p \sqrt{T}$ and $V_2(T) = \mu_p T + \sigma_p \sqrt{T}$. What is the range $[V_1, V_2]$ of typical fluctuations at time $T = 1$ for the investment of $w = $100,000 made above? Would you make this investment? Why? (Make a comparison with the returns you may get from this investment $\mu_p T \pm \sigma_p \sqrt{T}$ and what you would get from putting your money in a bank account paying a continuous compounding interest rate of $r = 4\%$. Is the return worth the risk?)

(b) Suppose the investor wants most to reduce the riskiness of the investment made in the two assets. Use your code to determine the optimal value of $\alpha$ which minimizes the variance of the portfolio for any return. This is equivalent of finding a zero of the function

$$\lambda_1(\alpha) = \frac{\partial \sigma_p^2}{\partial \alpha} = 2 \left(-w_1 \sigma_1^2 + (w_1 - w_2)\sigma_{1,2} + w_2 \sigma_2^2\right).$$ (3)

This can again be solved analytically, so check your code gives the correct result for $\alpha$. What are the weights $w_1$, $w_2$? What is the return $\mu_p$ of this optimal portfolio? What is the range of typical fluctuations $[V_1, V_2]$? Would you make this investment? Why?
(c) We shall now consider certain assets (such as a factory) which has an economy of scale so that the expected return may in fact increase as more resources are invested in the asset. Let us consider assets with returns

$$\mu_1(w_1) = 0.0005e^{3w_1}$$
$$\mu_2(w_2) = 0.07$$

and variances

$$\sigma^2_1(w_1) = e^{-3w_1}$$
$$\sigma^2_2(w_2) = 0.4.$$ 

The portfolio then has

$$\mu_p = w_1\mu_1(w_1) + w_2\mu_2(w_2)$$
$$\sigma^2_p = w_1^2\sigma^2_1(w_1) + 2w_1w_2\sigma_{1,2}(w_1, w_2) + w_2^2\sigma^2_2(w_2).$$

Let the assets have $\sigma_{1,2} = -0.01$.

Use that $w_1 = (1 - \alpha)$ and $w_2 = \alpha$ and your Matlab/Octave code to find the $\alpha$ which for a $w = $1,000,000 investment gives a portfolio with expected return $\mu_p = 0.05$. Allow $\alpha$ to vary in the range $[-1, 1]$, where negative weights correspond to going short on an asset. What are the weights $w_1, w_2$? What is the variance $\sigma^2_p$ of this portfolio? What is the range of typical fluctuations $[V_1, V_2]$? Would you make this investment? Why?

(d) Suppose that an investor wishes to invest $w = $1,000,000 in the least risky portfolio comprised of the two assets. Find the value of $\alpha$ which gives this portfolio. Allow $\alpha$ to vary in the range $[-1, 1]$, where negative weights correspond to going short on an asset. Use your Matlab/Octave code to find the zero of the derivative of the variance $\sigma^2_p$ of part (c), which in this case is given by:

$$\frac{\partial \sigma^2_p}{\partial \alpha} = \lambda_1(\alpha) + 3w_1^2\sigma^2_1.$$ 

What are the weights $w_1, w_2$? What is the expected return of this portfolio $\mu_p$? What is the range of typical fluctuations $[V_1, V_2]$? Would you make this investment? Why?