Midterm Exam:
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Introduction to Numerical Analysis, 104A
July 14th, 2009

Scoring:

Problem1: 2.5

Problem2: 2.5

Problem3: 2.5

Problem4: 2.5

Directions: Answer each question carefully and be sure to show all of your work. You are permitted to use a calculator but please be sure to show intermediate steps in your calculations. If you have any questions please feel free to ask.
**Problem 1:** Compute to four decimal places the absolute and relative errors when approximating \( p \) by the value \( p^* \). Also, state the number of significant digits.

a) \( p = \pi \) by \( p^* = 3.1416 \)

\[
\varepsilon_{abs} = |p^* - p| = 7.3464 \times 10^{-6}
\]

\[
\varepsilon_{rel} = \frac{|p^* - p|}{|p|} = 2.3354 \times 10^{-6}
\]

*Six significant digits.*

b) \( p = e^\pi \) by \( p^* = 1157/50 \)

\[
\varepsilon_{abs} = |p^* - p| = 6.9263 \times 10^{-4}
\]

\[
\varepsilon_{rel} = \frac{|p^* - p|}{|p|} = 2.9931 \times 10^{-5}
\]

*Five significant digits.*
Problem 2: In this problem we shall use $k$-digit-chopping to model the role of floating point arithmetic and the accuracy of numerical values computed using different algorithms. As a model problem, consider 2-digit-chopping for the number representation and arithmetic used to compute the numerical value $p^*$ which is suppose to approximate $p$. Consider each of the formulas given below. Specifically state: (i) the final numerical value, (ii) the number of significant digits in your final solution, and (iii) for this particular calculation of $p$, which formula is to be preferred (has smaller error).

Formula 1: $p_1^* = ((e^1 \cdot \pi + \sqrt{2} \cdot \pi) + e^1 \sqrt{8}) + 4$

Formula 2: $p_2^* = (\sqrt{8} + \pi) \cdot (\sqrt{2} + e^1)$

To high precision the solution is given by $p = 24.67107921715017$.

\[
\begin{align*}
\epsilon_1 & \rightarrow 2, 7, \pi \rightarrow 3, 1, \sqrt{2} \rightarrow 1, 4, \sqrt{8} \rightarrow 2, 8 \\
e^1 \cdot \pi & \rightarrow (2, 7) \cdot (3, 1) \rightarrow 8, 3 \\
\sqrt{2} & \rightarrow (1, 4) \cdot (3, 1) \rightarrow 4, 3 \\
e^1 \cdot \sqrt{8} & \rightarrow (2, 7) \cdot (2, 8) \rightarrow 7, 5 \\
(e^1 \cdot \pi + \sqrt{2} \cdot \pi) & \rightarrow (8, 3) + (4, 3) \rightarrow 12 \\
((e^1 \cdot \pi + \sqrt{2} \cdot \pi) + e^1 \cdot \sqrt{8}) & \rightarrow 12 + 7, 5 \rightarrow 19 \\

p_1^* & = 12, 3 \\
\varepsilon_{rel} & = \frac{|p_1^* - p|}{|p_1|} = 6, 7734 \times 10^{-2}, \text{ one significant digit}
\end{align*}
\]

\[
\begin{align*}
\sqrt{8} + \pi & \rightarrow (2, 8) + (3, 1) \rightarrow 5, 9 \\
\sqrt{2} + e^1 & \rightarrow (1, 4) + (2, 7) \rightarrow 4, 1 \\
(\sqrt{8} + \pi) \cdot (\sqrt{2} + e^1) & \rightarrow (5, 9) \cdot (4, 1) \rightarrow 4 \\

p_2^* & = 4 \\
\varepsilon_{rel} & = \frac{|p_2^* - p|}{|p_1|} = 2, 7962 \times 10^{-2}, \text{ two significant digits}
\end{align*}
\]

Formula 2 is more precise.
Problem 3: For the following fixed point iteration methods $x_{n+1} = g(x_n)$ determine the fixed point and the rate of convergence. In particular, state if the method converges at a rate which is linear, quadratic, or higher order.

a) $g(x) = x - \frac{1}{100}(x^3 - x)$, with $|x_0| < 1$.

Fixed Points: $x^* = 0, x^* = 1, x^* = -1$.

Let $e_k = x_k - x^*$ then.

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) = g'(x^*) (x_k - x^*) + \frac{1}{2} g''(\xi) (x_k - x^*)^2$$

$x^* = 0$: $g'(0) = 1 - \frac{1}{100} (3 \cdot 0 - 1) = 1 + \frac{1}{100} = \frac{101}{100} > 1$

$e_{k+1} = \left(\frac{101}{100}\right) e_k + \frac{1}{2} g''(\xi) e_k^2$, For $e_k$ small we see that $e_{k+1} \leq \left(\frac{101}{100}\right) e_k$ which shows that iteration does not converge to $x^* = 0$ (since $\frac{101}{100} > 1$).

$x^* = -1$: $g'(-1) = 1 + \frac{2}{100} = \frac{102}{100} > 1$, does not converge to this f. pt.

$x^* = 1$: $g'(1) = 1 - \frac{3}{100} = \frac{97}{100} < 1$, will converge provided $x_0$ suff. close to $x^*$, Rate of convergence is linear since $g'(1) \neq 0$.

b) $g(x) = x - \frac{(x^2 - 1)}{2x}$, with $|x_0| < 1$.

Fixed Points: $x^* = 1, x^* = -1$.

$g'(x) = 1 - \frac{2x}{2x} + \frac{(x^2 - 1)(x)}{(2x)^2} = \frac{(x^2 - 1)}{2x^2}$

$g''(x) = \frac{2x}{2x^2} - \frac{2(x^2 - 1)}{2x^2} = \frac{1}{x} - \frac{(x^2 - 1)}{x^3}$

$$e_{k+1} = g(x_k) - g(x^*) = g'(x^*) e_k + \frac{1}{2} g''(x^*) e_k^2 + \frac{1}{6} g'''(\xi) e_k^3$$

$x^* = 1$: $g'(1) = g'(1) = 0$, $g''(1) = 1$, for $e_k$ suff. small $e_{k+1} \leq \frac{1}{2} e_k^2$, so provided $x_0$ starts sufficiently close to $x^*$ the convergence is quadratic.

$x^* = -1$: (follows similarly).
Problem 4: For the given data points compute the Lagrange interpolating polynomial of degree three using the method of Divided Differences. State your answer in terms of a polynomial of the form \( P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \).

a) Consider the data given by \( (x_k, f(x_k)) : (-2, -2), (-1, -1), (1, -1), (2, -2) \).

\[
\begin{array}{cccccc}
X_k & F[X_k] & F[X_k, X_{k+1}] & F[X_k, X_{k+1}, X_{k+2}] & F[X_k, X_{k+1}, X_{k+2}, X_{k+3}] \\
-2 & -2 & 1 & & \\
-1 & -1 & & -1 & \\
1 & -1 & & -1 & 0 \\
2 & 4 & & & \\
\end{array}
\]

\[
P(x) = -2 + (x - x_0) + \frac{-1}{3} (x - x_0)(x - x_1)
\]

b) Consider the data given by \( (x_k, f(x_k)) : (-2, 4), (-1, 1), (1, 1), (2, 4) \).

\[
\begin{array}{cccccc}
X_k & F[X_k] & F[X_k, X_{k+1}] & F[X_k, X_{k+1}, X_{k+2}] & F[X_k, X_{k+1}, X_{k+2}, X_{k+3}] \\
-2 & 4 & -3 & & \\
-1 & 1 & & 0 & \\
1 & 1 & & 3 & \\
2 & 4 & & & \\
\end{array}
\]

\[
P(x) = 4 + -3(x - x_0) + (x - x_0)(x - x_1)
\]