Final Exam:
Professor: Paul J. Atzberger
Partial Differential Equations, 124A
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Scoring:

Problem 1: 2.5

Problem 2: 2.5

Problem 3: 2.5

Problem 4: 2.5

Total Score: 10.0

Directions: Please answer each question carefully. Be sure to show your work. If you have questions, please feel free to ask.
Problem 1:

a) Use method of characteristics to find the solution of

\[
\begin{align*}
3u_t(x,t) + 9u_x(x,t) &= 0, \quad -\infty < x < \infty, \quad t > 0 \\
u(x,0) &= \phi(x), \quad -\infty < x < \infty, \quad t = 0.
\end{align*}
\]

Let \( \overset{\circ}{\gamma}(s) = \begin{bmatrix} \gamma_1(s) \\ \gamma_2(s) \end{bmatrix} \) be a curve in the \((x,t)\)-plane with \( \overset{\circ}{\gamma}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \).

\[
\frac{\partial}{\partial s} u(\overset{\circ}{\gamma}(s)) = \gamma_1'(s)u_t + \gamma_2'(s)u_x = 3u_t + 9u_x = 0 \Rightarrow \begin{cases}
\gamma_1'(s) = 3s \\
\gamma_2'(s) = 9s + c_2 \end{cases}
\]

\( \overset{\circ}{\gamma}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0, \quad c_2 = x_0 \Rightarrow \begin{bmatrix} \gamma_1(s) \\ \gamma_2(s) \end{bmatrix} = \begin{bmatrix} 3s \\ 9s + x_0 \end{bmatrix} \).

We next find \( s \) and \( x_0 \) so this curve passes through the point \((x_1, x_1)\). This requires

\[ \overset{\circ}{\gamma}(s_1) = \begin{bmatrix} t_1 \\ x_1 \end{bmatrix} \Rightarrow \begin{cases} 3s_1 = t_1 \\
9s_1 + x_0 = x_1 \end{cases} \Rightarrow \begin{cases} s_1 = \frac{1}{3}t_1 \\
x_0 = x_1 - 3t_1 \end{cases} \]

Now since

\[
\frac{\partial}{\partial s} u(\overset{\circ}{\gamma}(s)) = 3u_t + 9u_x = 0,
\]

we have \( u(\overset{\circ}{\gamma}(s_1)) = u(\overset{\circ}{\gamma}(0)).\)

This implies

\[ u(t_1, x_1) = u(0, x_0) = \phi(x_0) = \phi(x_1 - 3t_1). \]

This gives the solution

\[ u(t, x) = \phi(x - 3t). \]
Problem 2:

a) Consider a solution of the Laplace Equation with Homogeneous Neumann Boundary Conditions

\[
\begin{align*}
\Delta w &= 0, \quad x \in \Omega \\
\frac{\partial w}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

Show any solution \( w \) of this Laplace Equation is a minimizer of the Dirichlet Energy given by

\[
E[w] = \int_\Omega |\nabla w|^2 \, dx.
\]  

(1)

The domain \( \Omega \) is assumed to be smooth and simply-connected  

**Hint:** Use Green’s First Identity to show that \( E[w] = 0 \)

**Green’s First Identity:**

\[
\int_\Omega w \Delta u \, dx = \int_{\partial \Omega} w \nabla u \cdot \mathbf{n} \, d\mathbf{x} - \int_\Omega \nabla w \cdot \nabla u \, dx
\]

In the case that \( w \) satisfies the PDE, we have

\[
E[w] = \int_\Omega \nabla w \cdot \nabla w \, dx = \int_{\partial \Omega} w \nabla w \cdot \mathbf{n} \, d\mathbf{x} - \int_\Omega w \Delta w \, d\mathbf{x} = 0.
\]
b) Consider two solutions \( u_1 \) and \( u_2 \) of the Poisson Equation with Neumann Boundary Conditions

\[
\begin{align*}
\Delta u &= g, \quad x \in \Omega \\
\frac{\partial u}{\partial n} &= h, \quad x \in \partial \Omega.
\end{align*}
\]

Use part (a) show that any two solutions differ only by a constant. In other words, show \( w = u_2 - u_1 = C_0 \) where \( C_0 \) is a constant. The domain \( \Omega \) is again assumed to be smooth and simply-connected. Hint: Consider what \( E[w] = 0 \) implies about the function \( w \).

Let \( w = u_2 - u_1 \) then \( w \) satisfies \( \begin{cases} \Delta w = 0, & x \in \Omega \\
\frac{\partial w}{\partial n} = 0, & x \in \partial \Omega \end{cases} \)

By part (a) this implies \( E[w] = 0 \).

\[
E[w] = \int_{\Omega} \nabla w \cdot \nabla w \, dx = 0 \Rightarrow \nabla w = 0 \Rightarrow w(x) = C_0,
\]

This shows \( u_2 = u_1 + C_0 \), so solutions only differ by a constant.
Problem 3:
Solve the Diffusion Equation on the interval $[-1, 1]$

\[
\begin{align*}
  u_t &= u_{xx}, & x \in (-1, 1), t > 0 \\
  u(-1, t) &= u(1, t) = 0, & t > 0 \\
  u(x, 0) &= \sin(\pi x) + \sin(7\pi x), & x \in (-1, 1), t = 0
\end{align*}
\]

Hint: Use Separation of Variables.

Consider $u$ expressed as $u(x, t) = X(x)T(t)$, then

\[
\begin{align*}
  u_t &= u_{xx} \\
  u_t = u_{xx} &\Rightarrow X(x)T'(t) = X''(x)T(t) &\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -c
\end{align*}
\]

\[
\Rightarrow T'(t) = -cT(t) \Rightarrow T(t) = \alpha e^{-ct}
\]

\[
X''(x) = -cX(x) \Rightarrow X(x) = \beta_1 e^{\sqrt{c}x} + \beta_2 e^{-\sqrt{c}x}.
\]

The boundary conditions require $u(\pm1, t) = X(\pm1)T(t) = 0$.

For $T(t) \neq 0$, we have $X$ must satisfy $X(\pm1) = 0$.

This implies $c < 0$, so we rewrite the constant as $c = -\gamma^2$.

\[
X(x) = \beta_1 e^{i\gamma x} + \beta_2 e^{-i\gamma x} = (\beta_1 + \beta_2) \cos(\gamma x) + i(\beta_1 - \beta_2) \sin(\gamma x).
\]

\[
X(-1) = [\beta_1 + \beta_2] \cos(-\gamma) + i(\beta_1 - \beta_2) \sin(-\gamma) = 0
\]

\[
X(1) = [\beta_1 + \beta_2] \cos(\gamma) + i(\beta_1 - \beta_2) \sin(\gamma) = 0
\]

\[
\Rightarrow (\beta_1 + \beta_2) \cos(\gamma) = 0 \Rightarrow \text{either } \beta_1 = \beta_2 = 0 \\
\text{or } (\beta_1 - \beta_2) \sin(\gamma) = 0 \Rightarrow \text{or } (\beta_1 - \beta_2) \sin(\gamma) = 0
\]

(i) $\cos(\gamma) = 0 \Rightarrow \gamma = \pi j, \ n \in \mathbb{Z}$, which gives solutions

\[
X_m(x) = A_m \cos((\pi j + m\pi)x), \ \text{where} \ \ c = -(\pi j + m\pi)^2
\]

(ii) $\sin(\gamma) = 0 \Rightarrow \gamma = \pi j, \ n \in \mathbb{Z}$, which gives solutions

\[
X_n(x) = B_n \sin((\pi j + n\pi)x), \ \text{where} \ \ c = -(\pi j + n\pi)^2
\]

This gives the general form for $u$

\[
u(x, t) = \sum_m A_m e^{-(\pi j + m\pi)^2 t} \cos((\pi j + m\pi)x) + \sum_n B_n e^{-(\pi j + n\pi)^2 t} \sin((\pi j + n\pi)x).
\]

Matching the initial conditions gives the solution

\[
\begin{align*}
  u(x, 0) &= \sin(\pi x) + \sin(7\pi x) \\
  u(x, t) &= e^{-\pi^2 t} \sin(\pi x) + e^{-49\pi^2 t} \sin(7\pi x).
\end{align*}
\]
Problem 4:
Solve the Wave Equation on $\mathbb{R}$

\[
\begin{align*}
  u_t &= u_{xx}, & x \in \mathbb{R}, t > 0 \\
  u(x, 0) &= \sin(\pi x), & x \in \mathbb{R}, t = 0 \\
  u_t(x, 0) &= 2xe^{-x^2}, & x \in \mathbb{R}, t = 0
\end{align*}
\]

The solution is given by

\[
u(x, t) = \frac{1}{2} \left[ \phi(x-ct) + \phi(x+ct) \right] + \frac{1}{x^2} \int_{x-ct}^{x+ct} \psi(s) \, ds
\]

For the given PDE, this gives the solution

\[
u(x, t) = \frac{1}{2} \left[ \sin(\pi(x-t)) + \sin(\pi(x+t)) \right] + \frac{1}{x^2} \int_{x-t}^{x+t} se^{-s^2} \, ds
\]

\[
= \frac{1}{2} \left[ \sin(\pi(x-t)) + \sin(\pi(x+t)) \right] - \frac{1}{2} \left[ e^{-(x+t)^2} - e^{-(x-t)^2} \right].
\]