5.3 ORTHOGONALITY AND GENERAL FOURIER SERIES

MATH 124B Solution Key HW 02

5.3 ORTHOGONALITY AND GENERAL FOURIER SERIES

1. (a) Find the real vectors that are orthogonal to the given vectors \((1, 1, 1)\) and \((1, -1, 0)\).
   (b) Choosing an answer to (a), expand the vector \((2, -3, 5)\) as a linear combination of these three mutually orthogonal vectors.

SOLUTION.

(a) Recall that the vector \((a, b, c)\) is orthogonal to both \((1, 1, 1)\) and \((1, -1, 0)\) if and only if it satisfies
   \[a + b + c = 0, \quad \text{and} \quad a - b = 0.\]
   One particular solution is \(a = 1, b = 1,\) and \(c = -2,\) respectively. The collection of all real vectors orthogonal to \((1, 1, 1)\) and \((1, -1, 0)\) is precisely the family of real scalar multiplies of \((a, b, c),\) i.e., the set
   \[\{t(1, 1, -2): t \in \mathbb{R}\}.\]

(b) Write
   \[(2, -3, 5) = \alpha(1, 1, -2) + \beta(1, 1, 1) + \gamma(1, -1, 0).\]
   Since the vectors on the right-hand-side of our equality are mutually orthogonal, we obtain
   \[(2, -3, 5) \cdot (1, 1, -2) = \alpha(1, 1, -2) \cdot (1, 1, -2),\]
   hence;
   \[-11 = 6\alpha, \quad \text{thus} \quad \alpha = -11/6.\]
   Therefore,
   \[\beta = 4/3, \quad \gamma = 5/2.\]

4. consider the problem \(u_t = ku_{xx}\) for \(0 < x < \ell,\) with the boundary conditions \(u(0, t) = U,\)
   \(u_x(\ell, t) = 0,\) and the initial condition \(u(x, 0) = 0,\) where \(U\) is a constant.
   (a) Find the solution in series form. Hint: Consider \(u(x, t) - U.\)
   (b) Using a direct argument, show that the series converges for \(t > 0.\)
   (c) If \(\epsilon\) is a given margin of error, estimate how long a time is required for the value \(u(\ell, t)\) at the endpoint to be approximated by the constant \(U\) with error \(\epsilon.\) Hint: It is an alternating series with first term \(U,\) so that the error is less than the next term.
(a) First, note that none of our available techniques apply to the given PDE since it fails
to vanish at the boundary condition \( x = 0 \). Following our hint, we recast the PDE by
writing
\[ v(x, t) = u(x, t) - U. \]
Hence we obtain the diffusion equation subject to the Dirichlet:
\[
\begin{cases}
  v_t = k v_{xx}, & 0 < x < \ell, 0 < t < \infty \\
  v(0, t) = 0, v_x(\ell, t) = 0 \\
  v(x, 0) = -U.
\end{cases}
\]
A slight modification of the argument presented in section 4.1 in our textbook makes
it plain that
\[ v(x, t) = \sum_{n=0}^{\infty} A_n e^{-(n+1/2)^2 \pi^2 kt/\ell^2} \sin(n + 1/2)\pi x/\ell. \]
The coefficients \( A_n \) are given by
\[ v(0, x) = -U = \sum_{n=0}^{\infty} A_n \sin(n + 1/2)\pi x/\ell. \]
By the orthogonality of these sine functions we obtain
\[
\int_{0}^{\ell} -U \sin(n + 1/2)\pi x/\ell = A_n \int_{0}^{\ell} \sin^2((n + 1/2)\pi x/\ell) \, dx = A_n \ell/2.
\]
Therefore,
\[
A_n = \frac{-2U}{\ell} \int_{0}^{\ell} \sin((n + 1/2)\pi x/\ell) \, dx
\]
\[ = 2U \frac{\cos((n + 1/2)\pi x/\ell)}{(n + 1/2)\pi} \bigg|_{0}^{\ell}
\]
\[ = 2U \left( \frac{\cos((n + 1/2)\pi)}{(n + 1/2)\pi} - \frac{1}{(n + 1/2)\pi} \right)
\]
\[ = 2U \left( \frac{-\sin n\pi}{(n + 1/2)\pi} - \frac{1}{(n + 1/2)\pi} \right)
\]
\[ = \frac{-2U}{(n + 1/2)\pi}.
\]
inasmuch \( \sin n\pi = 0 \) for all \( n \). To conclude, we obtain
\[ u(x, t) = U - v(x, t) = U - \frac{4U}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n + 1} e^{-(n+1/2)^2 \pi^2 kt/\ell^2} \sin(n + 1/2)\pi x/\ell. \]
5.3 ORTHOGONALITY AND GENERAL FOURIER SERIES

(b) Since fixed constants have no influence on convergence we may ignore both the initial term \( U \) and the scaling factor \( 4U/\pi \). We test for absolute convergence (which in turn implies the desired point-wise convergence) by invoking the comparison test. Moreover, we also make extensive use of the triangle inequality along with the bound \( |\sin y| \leq 1 \).

\[
\sum_{n=0}^{\infty} \left| \frac{1}{2n+1} e^{-(n+1/2)^2 \pi^2 kt^2} \sin(n+1/2)\pi x/\ell \right| \leq \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(n+1/2)^2 \pi^2 kt^2} \leq \sum_{n=0}^{\infty} e^{-(n+1/2)^2 \pi^2 kt^2} \leq \sum_{n=0}^{\infty} e^{-n^2 kt^2/2t^2} \sum_{n=0}^{\infty} e^{-n \pi^2 kt^2/2t^2} = e^\pi \sum_{n=0}^{\infty} \left( e^{-n \pi^2 kt^2/2t^2} \right)^n.
\]

The last series is a geometric series, which converges for \( t > 0 \) since

\[
0 < e^{-\pi^2 kt^2} < 1.
\]

(c) Following the hint we observe that at spacial position \( x = \ell \), the difference between the solution \( u(t, \ell) \) and \( U \) is given by

\[
-\frac{4U}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-(n+1/2)^2 \pi^2 kt^2}.
\]

This defines an alternating series, whose sum, in absolute value, is no bigger than the magnitude of the first term. That is,

\[
|u(\ell, t) - U| \leq \frac{4U}{\pi} e^{-\pi^2 kt/4t^2}.
\]

To ensure an error smaller than some fixed \( \varepsilon > 0 \), we solve for \( t \) in

\[
\frac{4|U|}{\pi} e^{-\pi^2 kt/4t^2} < \varepsilon.
\]

We find that

\[
-t < \frac{4\ell^2}{k \pi^2} \log \left( \frac{\pi \varepsilon}{4|U|} \right),
\]

or equivalently,

\[
t > \frac{4\ell^2}{k \pi^2} \log \left( \frac{4|U|}{\pi \varepsilon} \right).
\]
7. Show by direct integration that the eigenfunctions associated with the Robin BC's namely
\[ \phi_n(x) = \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x \quad \text{where} \quad \lambda_n = \beta_n^2, \]
are mutually orthogonal on \( 0 \leq x \leq \ell \), where \( \beta_n \) are the positive square roots of
\[
(\beta^2 - a_0 a_\ell) \tan \beta \ell = (a_0 + a_\ell) \beta. \tag{4.8}
\]

**SOLUTION.** We proceed by integrating directly, that is,
\[
I = \int_0^\ell \left( \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x \right) \left( \cos \beta_k x + \frac{a_0}{\beta_k} \sin \beta_k x \right) \, dx
= I_1 + I_2 + I_3 + I_4,
\]
where we use the shorthand notation
\[
I_1 = \int_0^\ell \cos \beta_n x \cos \beta_k x \, dx,
\]
\[
I_2 = \frac{a_0}{\beta_k} \int_0^\ell \cos \beta_n x \sin \beta_k x \, dx,
\]
\[
I_3 = \frac{a_0}{\beta_n} \int_0^\ell \sin \beta_n x \cos \beta_k x \, dx
\]
\[
I_4 = \frac{a_0^2}{\beta_n \beta_k} \int_0^\ell \sin \beta_n x \sin \beta_k x \, dx.
\]
After integrating by parts, we find:
\[
I_2 = \frac{a_0}{\beta_k} \int_0^\ell \cos \beta_n x \sin \beta_k x \, dx
= \frac{a_0}{\beta_n \beta_k} \sin \beta_n \ell \sin \beta_k \ell - \frac{a_0}{\beta_n} \int_0^\ell \sin \beta_n x \cos \beta_k x \, dx
= \frac{a_0}{\beta_n \beta_k} \sin \beta_n \ell \sin \beta_k \ell - I_3.
\]
Therefore, we have that
\[
I_2 + I_3 = \frac{a_0}{\beta_n \beta_k} \sin \beta_n \ell \sin \beta_k \ell. \tag{7.1}
\]
Next, we streamline the computation of \( I_4 \) by combining it with the computation of \( I_1 \). To do this, we again integrate by parts as follows

\[
I_4 = \frac{a_0^2}{\beta_n^2\beta_k^2} \int_0^\ell \sin \beta_n x \sin \beta_k x \, dx
\]

\[
= -\frac{a_0^2}{\beta_n^2\beta_k^2} \cos \beta_n \ell \sin \beta_k \ell + \frac{a_0^2}{\beta_n^2} \int_0^\ell \cos \beta_n x \cos \beta_k x \, dx
\]

\[
= -\frac{a_0^2}{\beta_n^2\beta_k^2} \cos \beta_n \ell \sin \beta_k \ell + \frac{a_0^2}{\beta_n^2} I_1.
\]

Armed with these results, we add together all the \( I_j \)s using this last line and (7.1) above we have that

\[
I = I_1 + I_2 + I_3 + I_4 = \frac{a_0}{\beta_n\beta_k} \sin \beta_n \ell \sin \beta_k \ell - \frac{a_0^2}{\beta_n^2\beta_k^2} \cos \beta_n \ell \sin \beta_k \ell + \left( 1 + \frac{a_0^2}{\beta_n^2} \right) I_1. \tag{7.2}
\]

To continue our calculations we recall the trigonometric identity:

\[
\cos \beta_n x \cos \beta_k x = \frac{1}{2} \left[ \cos(\beta_n + \beta_k)x + \cos(\beta_n - \beta_k)x \right].
\]

Thus we have that

\[
I_1 = \frac{1}{2} \int_0^\ell \left[ \cos(\beta_n + \beta_k)x + \cos(\beta_n - \beta_k)x \right] \, dx
\]

\[
= \frac{1}{2(\beta_n + \beta_k)} \sin(\beta_n + \beta_k)\ell + \frac{1}{2(\beta_n + \beta_k)} \sin(\beta_n + \beta_k)\ell
\]

\[
= I_{11} + I_{12}.
\]

To compute the last two terms, we employ the double angle identity for the sine function so that

\[
I_{11} = \frac{1}{2(\beta_n + \beta_k)} \left[ \sin \beta_n \ell \cos \beta_k \ell + \cos \beta_n \ell \sin \beta_k \ell \right]
\]

\[
I_{12} = \frac{1}{2(\beta_n - \beta_k)} \left[ \sin \beta_n \ell \cos \beta_k \ell - \cos \beta_n \ell \sin \beta_k \ell \right].
\]

Next, we notice that the identity \( (4.8) \) is equivalent to

\[
\sin \beta_n \ell = \frac{a_0 + a_\ell}{\beta_n^2} \beta_n \ell \cos \beta_n \ell, \tag{7.3}
\]

and similarly for \( \sin \beta_k \ell \). Please observe that is where we shall use the fact that \( \beta_n \) and \( \beta_k \) are generated by the Robin condition. Utilizing this last identity, and rationalizing all
denominators, we have that

\[
I_{11} + I_{12} = \frac{a_0 + a_\ell}{2(\beta_n^2 - \beta_k^2)} \cdot \frac{1}{(\beta_n^2 - a_0 a_\ell)(\beta_k^2 - a_0 a_\ell)} \cdot \cos \beta_n \ell \cos \beta_k \ell \\
\left[ (\beta_n - \beta_k)(\beta_k^2 - a_0 a_\ell) + (\beta_n - \beta_k)(\beta_n^2 - a_0 a_\ell) \beta_k \\
+ (\beta_n + \beta_k)(\beta_k^2 - a_0 a_\ell) \beta_n - (\beta_n + \beta_k)(\beta_n^2 - a_0 a_\ell) \beta_k \right].
\]

Now, a direct algebraic calculation involving Pascal’s triangle formula shows that

\[
\left[ (\beta_n - \beta_k)(\beta_k^2 - a_0 a_\ell) + (\beta_n - \beta_k)(\beta_n^2 - a_0 a_\ell) \beta_k \\
+ (\beta_n + \beta_k)(\beta_k^2 - a_0 a_\ell) \beta_n - (\beta_n + \beta_k)(\beta_n^2 - a_0 a_\ell) \beta_k \right]
= -2a_0 a_\ell (\beta_n^2 - \beta_k^2).
\]

Therefore, the previous computation simplifies considerably to the following:

\[
I_1 = I_{11} + I_{12} = -(a_0 + a_\ell) \cos \beta_n \ell \cos \beta_k \ell \cdot \frac{a_0 a_\ell}{(\beta_n^2 - a_0 a_\ell)(\beta_k^2 - a_0 a_\ell)}.
\] (7.4)

Finally, if we substitute (7.4) for \(I_1\) in (7.2) above, and if we furthermore trade out the sines for cosines in the first two terms on the right-hand-side of that expression using (7.3), we have that

\[
I = (a_0 + a_\ell) \cos \beta_n \ell \cos \beta_k \ell \cdot \left[ \frac{a_0(a_0 + a_\ell)}{(\beta_n^2 - a_0 a_\ell)(\beta_k^2 - a_0 a_\ell)} \\
- \frac{a_0^2(\beta_n - a_0 a_\ell)}{\beta_n^2(\beta_k^2 - a_0 a_\ell)(\beta_k^2 - a_0 a_\ell)} - \left( 1 + \frac{a_0^2}{\beta_n^2} \right) \frac{a_0 a_\ell}{(\beta_n^2 - a_0 a_\ell)(\beta_k^2 - a_0 a_\ell)} \right] 
\].

After a moment’s thought we now realize that the three terms inside the bracket on the right-hand-side of this last expression sum to zero. In other words, the desired conclusion follows and \(I = 0\) as desired. ■

9. Show that the boundary conditions

\[
X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)
\]
on an interval \(a \leq x \leq b\) are symmetric if and only if \(\alpha \delta - \beta \gamma = 1\).

**SOLUTION.** Suppose that \(f\) and \(g\) both satisfy the given boundary conditions. That is,

\[
\begin{align*}
    f(b) &= \alpha f(a) + \beta f'(a) \\
    f'(b) &= \gamma f(a) + \delta f'(a) \\
    g(b) &= \alpha g(a) + \beta g'(a) \\
    g'(b) &= \gamma g(a) + \delta g'(a)
\end{align*}
\]
A direct calculation makes it plain that
\[
-f'g' + f'g \bigg|_a^b = -f(b)g'(b) + f'(b)g(b) + f(a)g'(a) - f'(a)g(a)
\]

\[
= -\left[ \alpha f(a) + \beta f'(a) \right] \left[ \gamma g(a) + \delta g'(a) \right]
\]

\[
+ \left[ \gamma f(a) + \delta f'(a) \right] \left[ \alpha g(a) + \beta g'(a) \right]
\]

\[
+ f(a)g'(a) - f'(a)g(a)
\]

\[
= -\alpha \gamma f(a)g(a) - \alpha \delta f(a)g'(a)
\]

\[
- \beta \gamma f'(a)g(a) + \beta \delta f'(a)g'(a)
\]

\[
+ a \gamma f(a)g(a) + \beta \gamma f(a)g'(a)
\]

\[
+ a \delta f'(a)g(a) + \beta \delta f'(a)g'(a)
\]

\[
+ f(a)g'(a) - f'(a)g(a)
\]

\[
= -\alpha \delta f(a)g'(a) - \beta \gamma f'(a)g(a) + \beta \gamma f(a)g'(a)
\]

\[
+ a \delta f'(a)g(a) + f(a)g'(a) - f'(a)g(a)
\]

\[
= (\beta \gamma - \alpha \delta + 1)f(a)g'(a) + (a \delta - \beta \gamma - 1)f'(a)g(a)
\]

Hence; it follows that we are in a symmetric case if and only if
\[
\beta \gamma - \alpha \delta + 1 = a \delta - \beta \gamma - 1 = 0,
\]

or equivalently, if and only if
\[
a \delta - \beta \gamma = 1
\]
as required.

12. Prove Green’s first identity\(^1\) For every pair of functions \(f(x), g(x)\) on \((a, b)\).

\[
\int_{a}^{b} f''(x)g(x) \, dx = \int_{a}^{b} f'(x)g'(x) \, dx + f'g \bigg|_{a}^{b}.
\]

**SOLUTION.** Integrating by parts we obtain the identity
\[
\int_{a}^{b} f''(x)g(x) \, dx = [f'(x)g(x)]' - f'(x)g'(x).
\]

An application of the fundamental theorem of calculus on the first term on the right-hand-side now makes the desired conclusion plain.
6.1 Laplace’s Equation

12. Check the validity of the maximum principle for the harmonic function $(1-x^2-y^2)/(1-2x+x^2+y^2)$ in the disk $\overline{D} = \{x^2 + y^2 \leq 1\}$. Explain.

**SOLUTION.** A routine calculation makes it plain that

$$f(x, y) = \frac{1-x^2-y^2}{1-2x+x^2+y^2} = \frac{1-(x^2+y^2)}{(x-1)^2+y^2}$$

is indeed harmonic on $D$. Moreover, on the boundary, say

$$\text{bdy } D = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\},$$

we see that $f = 0$, which is majorized by the interior values, except at the point $(1, 0)$ where $f$ is not defined. Nevertheless, this does not contradict the conclusion of the maximum principle because our function $f$ fails to be continuous on the closure $\overline{D}$. ■

13. A function $u(x)$ is subharmonic in $D$ if $\Delta u \geq 0$ in $D$. Prove that its maximum value is attained on $\text{bdy } D$. **Remark:** Note that this is not true for the minimum value.

**SOLUTION.** We proceed by mimicking the proof of the maximum principle in our textbook with minor modification. Fix $\epsilon > 0$ and put

$$v(x) = u(x) + \epsilon|x|^2.$$ 

A direct calculation makes it plain that

$$\Delta v = \Delta u + \epsilon \Delta (x^2 + y^2) \geq 0 + 4\epsilon > 0,$$

But necessarily, $\Delta v \leq 0$ at an interior maximum, by the 2nd derivative test for optimization from elementary calculus. Therefore, $v$ cannot admit an interior maximum in $D$. Now, since $v$ is continuous on a closed and bounded set $\overline{D}$ then the extreme value theorem applies and $v$ attains a maximum in $\overline{D}$. Say the maximum of $v$ is attained at $x_o \in \text{bdy } D$, then for all $x \in D$, we have

$$u(x) \leq v(x) \leq v(x_o) = u(x_o) + \epsilon|x_o|^2 \leq \max_{\text{bdy } D} u + \epsilon \ell^2,$$

where $\ell$ denotes the greatest distance from $\text{bdy } D$ to the origin. Since our choice of $\epsilon$ may be arbitrarily small it follows that

$$u(x) \leq \max_{\text{bdy } D} u \quad \text{for all} \quad x \in D.$$

Now, this maximum is attained at some $x_M \in \text{bdy } D$ by the extreme value theorem. So

$$u(x) \leq u(x_M) \quad \text{for all} \quad x \in \overline{D}.$$
Since our inequality is not strict we cannot applying the previous argument to \(-u\), as is the case with the original proof. It now remains to prove the absence of interior extrema. To see this, let \(M\) denote the maximum value of \(u\) on \(\overline{D}\) as furnished by the extreme value theorem. Then

\[
u(x) \leq u(x_M) = M \quad \text{for all} \quad x \in D.
\]

Say \(u(x_0) = M\) for some \(x_0 \in \text{interior } D\) and let \(y\) be any other point in \(D\). Since \(D\) is open and connected in \(\mathbb{R}^2\) it must be the case that \(D\) is polygonally path-connected.\(^2\) Let \(\Gamma\) denote the polygonal path in \(D\) that joins \(x_0\) to \(y\) and put

\[
d = \min\{\|x - y\|: x \in D, y \in \text{bdy } D}\.
\]

Additionally, there exists a sequence of balls \(B_r(x_i)\) for \(i = 0, 1, 2, \ldots, n\) with \(x_i \in \Gamma\) satisfying \(r \leq d\), and \(x_{i+1} \in B_r(x_i)\) where \(y = x_n\). Since our function \(u\) is harmonic it must satisfy the mean-value property. Now, since \(u \leq M\) in \(B_r(x_0)\) and \(u(x_0) = M\) it follows that

\[
M = u(x_0) = \text{average on circle} \leq M,
\]

hence; \(u \equiv M\) on \(B_r(x_0)\). But then \(u \equiv M\) on \(B_r(x_i)\) for all \(i\) since \(x_{i+1} \in B_r(x_i)\). In particular, \(u(y) = M\). Since our choice of \(y\) was arbitrary it follows that \(u \equiv M\) on \(D\) and we have established the contrapositive.\(^3\)

---

\(^2\)This is an advanced result from point-set topology.

\(^3\)This was a particularly difficult exercise. Don’t be discouraged if you have problems following the arguments on a first read.