7.1 GREEN’S FIRST IDENTITY

MATH 124B Solution Key HW 05

7.1 GREEN’S FIRST IDENTITY

1. Derive the 3-dimensional maximum principle from the mean value property.

**SOLUTION.** We aim to prove that if \( u \) is harmonic in the bounded set \( D \subset \mathbb{R}^3 \) and \( u \) is continuous on \( \overline{D} = D \cup \text{bdy } D \) then the maximum and minimum of \( u \) occurs on bdy \( D \) and nowhere else, provided that \( u \) is not constant. We proceed by proving the contrapositive. That is, if \( u \) admits an interior extremum then \( u \) must be constant.

Since \( u \) is continuous on the closed, bounded set \( \overline{D} \), then \( u \) must attain both its maximum, say \( M \), and its minimum, say \( m \), somewhere in \( \overline{D} \). Suppose that \( u \) attains an interior extremum at the point \( x_0 \in D \) and let \( x^* \) be any other point in \( D \). Since \( D \) is open and connected in \( \mathbb{R}^3 \) it follows that \( D \) is polygonally path connected.\(^1\) Let \( \Gamma \) be the polygonal path in \( D \) joining \( x_0 \) and \( x^* \) and let \( \delta \) be the shortest distance separating \( \Gamma \) and bdy \( D \).\(^2\)

\[
\delta = \min\{\|x - y\| : x \in \Gamma, x \in \text{bdy } D\}.
\]

By the compactness of \( D \) there exists a sequence of balls \( B_R(x_i) \), for \( i = 0, 1, \ldots, n \), with \( x_i \) on \( \Gamma \), satisfying \( R \leq \delta \), \( x_{i+1} \in B_R(x_i) \), \( x^* \in B_R(x_n) \). By the mean-value property we see that \( u \) is identically equal to \( M \) in each \( B_R(x_i) \), \( i = 0, 1, 2, \ldots n \); hence,

\[
u(x^*) = M.
\]

Since our choice of \( x^* \) was completely arbitrary, \( u \) must be identically equal to \( M \) throughout \( D \) and, by continuity of \( u \), throughout all of \( \overline{D} \). Indeed, this proves that if \( u \) is not constant in \( D \), then \( u \) attains its maximum value on the boundary of \( D \) and nowhere else. Similarly, this argument applied to \(-u\) establishes that if \( u \) is non-constant, it can attain its minimum value on bdy \( D \) and nowhere else. \(\blacksquare\)

2. Prove the uniqueness up to constant of the Neumann problem using the energy method.

**SOLUTION.** Suppose that \( u_1 \) and \( u_2 \) solve the Neumann problem for harmonic functions, that is,

\[
\begin{align*}
\Delta u &= 0 & \text{in } & D \\
\frac{\partial u}{\partial n} &= g(x) & \text{on } & \text{bdy } D.
\end{align*}
\]

\(^1\)This is an elementary result from point-set topology in Math 118A
\(^2\)The existence of such a number \( \delta > 0 \) follows by the continuity of the norm and the compactness of \( \overline{D} \). Again, a result in math 118A.
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Put \( w = u_1 - u_2 \), then \( w \) solves the homogeneous Neumann problem

\[
\begin{align*}
\Delta w &= 0 \quad \text{in } D \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial D.
\end{align*}
\]

An application of Green’s first identity makes it plain that

\[
\int_{\partial D} w \frac{\partial w}{\partial n} \, dS = \int_D \nabla w \cdot \nabla w \, d\mathbf{x} + \int_D w \Delta w \, d\mathbf{x}.
\]

Since \( \frac{\partial w}{\partial n} = 0 \) on \( \partial D \) and \( \Delta w = 0 \) in \( D \), we see that

\[
0 = \int_D \nabla w \cdot \nabla w \, d\mathbf{x} = \int_D |\nabla w|^2 \, d\mathbf{x}.
\]

Since the integrand \( |\nabla w|^2 \geq 0 \), it follows by the vanishing theorem that \( |\nabla w|^2 \equiv 0 \) in \( D \), that is, \( \nabla w \equiv 0 \) in \( D \) and we deduce that \( w \) is constant in \( D \), say \( w \equiv C \). By our construction we see that \( u_1 = u_2 + C \) in \( D \) and we have established uniqueness of the Neumann problem up to a constant as required.

5. Prove Dirichlet’s principle for the Neumann boundary condition. It asserts that among all real-valued functions \( w(x) \) on \( D \) the quantity

\[
E[w] = \frac{1}{2} \int_D |\nabla w|^2 \, d\mathbf{x} - \int_{\partial D} hw \, dS
\]

is the smallest for \( w = u \), where \( u \) is the solution of the Neumann problem

\[
-\Delta u = 0 \quad \text{in } D, \quad \frac{\partial u}{\partial n} = h(x) \quad \text{on } \partial D.
\]

It is required to assume that the average of the given function \( h(x) \) is zero (by Exercise 6.1.11).

Notice three features of this principle

(i) There is no constraint at all on the trail functions \( w(x) \).

(ii) The function \( h(x) \) appears in the energy.

(iii) The functional \( E[w] \) does not change if a constant is added to \( w(x) \).

**Hint:** Follow the method in Section 7.1.
**SOLUTION.** For the sake of completeness we present the solution to Exercise 6.1.11. Suppose that \( u \) solves the Neumann problem

\[
\begin{aligned}
\Delta u &= f \quad \text{in} \quad D \\
\frac{\partial u}{\partial n} &= g \quad \text{on} \quad \text{bdy} \, D.
\end{aligned}
\]

By taking \( v \equiv 1 \) in Green's 1st identity we see that

\[
\iint_{\text{bdy} \, D} \frac{\partial}{\partial n} u \, dS = \iiint_{D} \Delta u \, d\mathbf{x}.
\]

Indeed, this imposes a necessary condition in order for the Neumann problem to be well-posed. More specifically, it must be the case that

\[
\iint_{\text{bdy} \, D} g(\mathbf{x}) \, dS = \iiint_{D} f(\mathbf{x}) \, d\mathbf{x}.
\]

Armed with this result we now return to our original problem and see that it must be the case that for \( f(\mathbf{x}) = 0 \)

\[
\iint_{\text{bdy} \, D} h(\mathbf{x}) \, dS = \iiint_{\text{bdy} \, D} \frac{\partial u}{\partial n} \, dS = 0.
\]

In other words, the average of the given function \( h(\mathbf{x}) \) is zero. By Exercise 2, the solution to the well-posed Neumann problem is unique up to a constant. Fix a solution, say \( u \).
and let \( w \) be any other trial function. By putting \( v = u - w \) and we see that

\[
E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, d\mathbf{x} - \iint_{\partial D} hw \, dS
\]

\[
= \frac{1}{2} \iiint_D |\nabla u - \nabla v|^2 \, d\mathbf{x} - \iint_{\partial D} h(u - v) \, dS
\]

\[
= \frac{1}{2} \iiint_D (\nabla u \cdot \nabla u - 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v) \, d\mathbf{x} - \iint_{\partial D} h(u - v) \, dS
\]

\[
= \frac{1}{2} \iiint_D |\nabla u|^2 \, d\mathbf{x} - \iint_{\partial D} h(u - v) \, dS
\]

where we invoke Green’s 1st identity for the last line. Now, since \( \Delta u = 0 \) in \( D \) we obtain that

\[
E[w] - E[u] = \frac{1}{2} \iiint_D |\nabla v|^2 \, d\mathbf{x}.
\]

Since the integrant \( |\nabla v|^2 \geq 0 \) it follows that

\[
E[w] - E[u] \geq 0,
\]

hence;

\[
E[w] \geq E[u]
\]

and the desired conclusion follows.

6. Let \( A \) and \( B \) be two disjoint bounded spatial domains, and let \( D \) be their exterior. So \( \partial D = \partial A \cup \partial B \). Consider a harmonic function \( u(x) \) in \( D \) that tends to zero at infinity, which is \textit{constant} on \( \partial A \) and \textit{constant} on \( \partial B \), and which satisfies

\[
\iint_{\partial A} \frac{\partial u}{\partial n} \, dS = Q > 0 \quad \text{and} \quad \iint_{\partial B} \frac{\partial u}{\partial n} \, dS = 0.
\]

\textit{Interpretation:} The harmonic function \( u(x) \) is the electrostatic potential of two conductors, \( A \) and \( B \); \( Q \) is the charge on \( A \), while \( B \) is uncharged.
(a) Show that the solution is unique. **Hint:** Use the Hopf maximum principle.

(b) Show that \( u \geq 0 \) in \( D \). **Hint:** If not, then \( u(x) \) has a negative minimum. Use the Hopf principle again.

(c) Show that \( u > 0 \) in \( D \).

**SOLUTION.** First, we recall the Hopf form of the maximum principle as presented in Exercise 12(a) on page 177 of our textbook.

If \( u(x) \) is a non-constant harmonic function in a connected plane domain \( D \) with a smooth boundary that has a maximum at \( x_0 \) (necessarily on the boundary by the strong maximum principle), then \( \partial u / \partial n > 0 \) at \( x_0 \) where \( n \) is the unit outward normal vector.

(a) The Hopf principle is pis aller since its proof is outside the scope of this class. Accordingly, we present an alternative proof. Suppose that \( u_1 \) and \( u_2 \) solve our PDE of interest, namely,

\[
\begin{align*}
\Delta u &= 0, \quad \text{in } D, \\
u &= a, \quad \text{on bdy } A, \\
u &= b, \quad \text{on bdy } B, \\
\int_{\text{bdy } A} \frac{\partial u}{\partial n} \, dS &= Q > 0 \quad \text{and} \quad \int_{\text{bdy } B} \frac{\partial u}{\partial n} \, dS = 0 \\
u(x) &\to 0, \quad \text{uniformly, as } |x| \to \infty.
\end{align*}
\]

Put \( w = u_1 - u_2 \), then \( w \) solves

\[
\begin{align*}
\Delta w &= 0, \quad \text{in } D, \\
w &= 0, \quad \text{on bdy } D = \text{bdy } A \cup \text{bdy } B, \\
\int_{\text{bdy } D} \frac{\partial w}{\partial n} \, dS &= \int_{\text{bdy } A} \frac{\partial w}{\partial n} \, dS = \int_{\text{bdy } B} \frac{\partial w}{\partial n} \, dS = 0 \\
w(x) &\to 0, \quad \text{uniformly, as } |x| \to \infty.
\end{align*}
\]

Let \( x_0 \) be an arbitrary point in \( D \). Since \( w(x) \to 0 \) as \( |x| \to \infty \), it follows that given \( \varepsilon > 0 \), the radius \( R \) can be chosen sufficiently large so that \( |x_0| < R \), and \( |w(x)| < \varepsilon \) for \( |x| \geq R \). In the bounded region \( D_R \) defined by the intersection of \( D \) and the ball \( |x| < R \), the maximum principle applies. Since the boundary of \( D_R \) consists partly of bdy \( D \) and partly of the surface of the sphere \( |x| = R \), and since \( w \equiv 0 \) on bdy \( D \) while \( |w| < \varepsilon \) on \( |x| = R \), it follows that \( |w(x)| < \varepsilon \) throughout \( D_R \). In particular,
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|w(x₀)| < ε. Since our choice of ε may be arbitrarily small it follows that |w(x₀)| = 0, hence w(x₀) = 0. Also, since the choice of x₀ ∈ D was arbitrary we see that w ≡ 0, or u₁ = u₂ in D and we have established uniqueness.

(b) First note that it cannot be the case that u is constant, otherwise u ≡ 0 since u vanishes at infinity. More specifically, ∇u ≡ 0, so n · ∇u = ∂u/∂n ≡ 0, which in turn contradicts the hypothesis that the charge on A is positive.

Second, when u is assumed to be non-constant we argue by means of a contradiction. Suppose that there exist a point x∗ ∈ D such that u(x∗) < 0. By applying the Hopf principle to −u we see that the minimum of u, that is, the maximum of −u is attained on bdy D and nowhere else, say at x₀, and −∂u/∂n(x₀) > 0, or equivalently ∂u/∂n(x₀) < 0. If the minimum is attained on bdy A, then the condition

\[ \int_{\partial D} \frac{\partial u}{\partial n} dS = Q > 0, \]

makes it plain that it cannot be the case that ∂u/∂n is negative on all of bdy A. More specifically, there must exist a point x_A ∈ bdy A such that ∂u/∂n(x_A) > 0, but then when moving outward along the normal directional derive it must be the case that the value of u is decreasing as we move away from the boundary of A into our domain D, inasmuch u is constant on bdy A. But this contradicts our assumption that the minimum value of u is attained on bdy A and nowhere else. Conversely, if the minimum of u is attained on bdy B, then the condition

\[ \int_{\partial B} \frac{\partial u}{\partial n} dS = 0, \]

along with the fact that ∂u/∂n ≠ 0 on bdy B makes it plain that there must exist a point x_B ∈ bdy B such that ∂u/∂n(x_B) > 0, but then again we arrive at a contradiction. Therefore it follows that u ≥ 0 on D.

(c) We argue by means of a contradiction. Suppose that there exists a point x∗ ∈ D where u(x∗) = 0. Since u cannot be constant it follows by the continuity of u there exist a small enough neighborhood along x∗ where u is negative. By applying the argument from part (b) to a point in such a neighborhood we arrive at yet another contradiction.

\[ (\text{Rayleigh-Ritz approximation to the harmonic function } u \text{ in } D \text{ with } u = h \text{ on bdy } D). \]

Let w₀, w₁, . . ., wₙ be arbitrary functions such that w₀ = h on bdy D and w₁ = ··· = wₙ = 0 on bdy D. The problem is to find constants c₁, c₂, . . ., cₙ so that

\[ w₀ + c₁w₁ + ··· + cₙwₙ \text{ had the least possible energy.} \]

Show that the constants must solve the linear system

\[ \sum_{k=1}^{n} (\nabla w_j, \nabla w_k)c_k = -(\nabla w₀, \nabla w_j) \text{ for } j = 1, 2, \ldots, n. \]
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**SOLUTION.** First, recall the inner-product notation

\[(\nabla w_j, \nabla w_k) = \iiint_D \nabla w_j \cdot \nabla w_k \, dx = (\nabla w_k, \nabla w_j)\]

and put

\[w = w_0 + c_1 w_1 + \cdots + c_n w_n = w_0 + \sum_{j=1}^n c_j w_j,\]

where \(w_0, c_j,\) and \(w_j\) have the properties described in the statement of our problem for \(j = 1, 2, \ldots, n.\) It follows that the energy is given by

\[E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, dx\]

which minimize \(E,\) the partial derivative of \(E\) with respect to each \(c_k\) must vanish. In other words, for each \(k = 1, 2, \ldots, n,\) it follows that

\[\frac{\partial}{\partial c_k} E[w] = 0.\]

More specifically,

\[0 = (\nabla w_0, \nabla w_k) + (\nabla w_k, \nabla w_k) c_k + \sum_{j \neq k} (\nabla w_k, \nabla w_j) c_j\]

\[= (\nabla w_0, \nabla w_k) + \sum_{j=1}^n (\nabla w_k, \nabla w_j) c_j,\]
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hence;

\[ \sum_{j=1}^{n} (\nabla w_k, \nabla w_j)c_j = -(\nabla w_0, \nabla w_k) \quad \text{for} \quad k = 1, 2, \ldots, n, \]

as required. \hfill \blacksquare