MATH 124B Solution Key HW 06

7.2 GREEN'S SECOND IDENTITY

1. Derive the representation formula for harmonic functions (7.2.5) in two dimensions.

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\int_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\log|\mathbf{x}-\mathbf{x}_0|) - \frac{\partial u}{\partial n}\log|\mathbf{x}-\mathbf{x}_0|\right] ds.$$

SOLUTION. Let $\mathbf{x}_0$ be any point in the two-dimensional region D. A direct calculation makes it plain that the function $v(\mathbf{x}) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0|$ satisfies the identity

$$\Delta v = \delta(x - \mathbf{x}_0), \tag{7.1}$$

where the Dirac delta satisfies the condition

$$\iint_D u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)\, d\,\mathbf{x} = u(\mathbf{x}_0).$$

Substituting (7.1) into Green's second identity along with the fact that $\Delta u = 0$ in D makes it plain that

$$\iint_D (u\Delta v - v\Delta u)\, d\,\mathbf{x} = \int_{\text{bdy } D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

takes the form

$$\begin{aligned}
u(\mathbf{x}_0) &= \iint_D u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)\, d\,\mathbf{x} \\
&= \iint_D u\Delta v\, d\,\mathbf{x} \\
&= \int_{\text{bdy } D}\left(u\frac{\partial u}{\partial n} - v\frac{\partial u}{\partial n}\right) ds \\
&= \int_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0|\right) - \frac{\partial u}{\partial n}\left(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0|\right)\right] ds \\
&= \frac{1}{2\pi}\int_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\log|\mathbf{x}-\mathbf{x}_0|) - \frac{\partial u}{\partial n}\log|\mathbf{x}-\mathbf{x}_0|\right] ds,
\end{aligned}$$

as required. ■

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2. Let $\phi(\mathbf{x})$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.

SOLUTION. Let S denote our sphere of interest and notice that bdy S is both the boundary of S and $\mathbb{R}^3\backslash S$. This seemingly trivial observation is absolutely crucial in our proceeding calculation. Since ϕ vanishes outside of S it follows that $\Delta\phi = 0$ in $\mathbb{R}^3\backslash S$. More specifically, the representation formula applies and

$$\begin{aligned}\phi(\mathbf{0}) &= \iint_{\text{bdy }\mathbb{R}^3\backslash S}\left[-\phi(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|}\frac{\partial\phi}{\partial n}\right]\frac{dS}{4\pi}\\ &= \frac{1}{4\pi}\iint_{\text{bdy }\mathbb{R}^3\backslash S}\left[-\phi(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|}\frac{\partial\phi}{\partial n}\right]dS.\end{aligned}$$

An application of Green's second identity makes it plain that

$$\begin{aligned}\phi(\mathbf{0}) &= \frac{1}{4\pi}\iint_{\text{bdy }\mathbb{R}^3\backslash S}\left[-\phi(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|}\frac{\partial\phi}{\partial n}\right]dS\\ &= \frac{1}{4\pi}\iiint_S\left(\phi\Delta\left(\frac{1}{|\mathbf{x}|}\right)-\frac{1}{|\mathbf{x}|}\Delta\phi\right)d\mathbf{x}\\ &= \frac{1}{4\pi}\iiint_S\left(\phi\delta(\mathbf{x})-\frac{1}{|\mathbf{x}|}\Delta\phi\right)d\mathbf{x},\end{aligned}$$

where δ denotes the familiar Dirac delta. If $\mathbf{0}\notin S$ then $\delta(\mathbf{x})\equiv 0$ on S. Hence;

$$\phi(\mathbf{0}) = -\iiint\frac{1}{|\mathbf{x}|}\Delta\phi\frac{d\mathbf{x}}{4\pi}.$$

as required. ■

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3. Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and $\mathbf{x}_0$ its center in the representation formula (1).

SOLUTION. Fix $\mathbf{x}_0\in\mathbb{R}^3$ and suppose that u is harmonic in the ball of radius a centered at $\mathbf{x}_0$, say $D = B(\mathbf{x}_0; a)$. An application of the representation formula makes it plain that

$$u(\mathbf{x}_0) = -\frac{1}{4\pi}\iint_{\text{bdy }D}u\frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}dS + \frac{1}{4\pi}\iint_{\text{bdy }D}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}dS.$$

Put $r = |\mathbf{x}-\mathbf{x}_0|$, then in polar coordinates on bdy D we have

$$\frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{x}_0|} = \frac{\partial}{\partial r}\frac{1}{r} = -\frac{1}{r^2},$$

so our first integral becomes

$$
\begin{aligned}
-\frac{1}{4\pi}\iint_{\text{bdy } D} u\frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,dS &= -\frac{1}{4\pi}\iint_{\text{bdy } D} u\left(-\frac{1}{r^2}\right)\,dS \\
&= -\frac{1}{4\pi}\iint_{\text{bdy } D} u\left(-\frac{1}{a^2}\right)\,dS \\
&= \frac{1}{4\pi a^2}\iint_{\text{bdy } D} u\,dS.
\end{aligned}
$$

An application of Green's first identity with $v \equiv 1$ allows us to rewrite our second integral as

$$
\begin{aligned}
\frac{1}{4\pi}\iint_{\text{bdy } D} \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\,dS &= \frac{1}{4\pi}\iint_{\text{bdy } D} \frac{1}{r}\frac{\partial u}{\partial n}\,dS \\
&= \frac{1}{4\pi}\iint_{\text{bdy } D} \frac{1}{a}\frac{\partial u}{\partial n}\,dS \\
&= \frac{1}{4\pi a}\iint_{\text{bdy } D} \frac{\partial u}{\partial n}\,dS \\
&= \frac{1}{4\pi a}\iiint_{D} \Delta u\,d\mathbf{x} = 0,
\end{aligned}
$$

since $\Delta u = 0$ in D. Putting all of this together we obtain

$$
u(\mathbf{x}_0) = \frac{1}{4\pi a^2}\iint_{\text{bdy } D} u\,dS = \frac{1}{\text{surface area (bdy } D)}\iint_{\text{bdy } D} u\,dS,
$$

as desired. ■