Homework 3
Introduction to Numerical Analysis
Professor: Paul J. Atzberger
Due: Thursday, February 1st
All homeworks are to be turned in at the beginning of class.

Problems:

1) The price of an option \( p(s_0) \) maturing at time \( T \) when the spot price is \( s_0 \) at time 0 can be determined from Black-Scholes-Merton pricing theory. The Black-Scholes-Merton option price formula can be expressed as:

\[
p(s_0) = e^{-rT} \int_{-\infty}^{\infty} V\left( s_0 \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} \cdot x \right] \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\]

where \( V(S) \) is the payoff of the option at maturity at stock price \( S \).

This can be viewed as an expectation over "random stock prices" with the assumption that the stock price \( S_T \) at time \( T \) having volatility \( \sigma \) obeys a lognormal distribution. More specifically, samples of the stock price at time \( T \) can be generated from the normal by using the expression:

\[
S_T = s_0 \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} X \right]
\]

where \( X \) is a sample of a standard normal random variable. Thus substituting \( S_T \) in the expression above we obtain:

\[
p(s_0) = e^{-rT} \int_{-\infty}^{\infty} V(S_T(x)) \rho(x) dx = e^{-rT} E^{s_0}[V(S_T)]
\]

where \( \rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). This puts the pricing formula in the familiar form to which we can apply our Monte-Carlo methods. For the following problems use your Monte-Carlo codes to estimate the price of the given option.

(a) Estimate the price of the call option by generating \( N = 10,000 \) samples of the stock price \( S_T \) at maturity time \( T = 100 \). Use the parameter values \( r = 0.05 \), \( \sigma = 0.20 \), \( s_0 = 10 \), \( K = 10 \). The payoff of the call option is given by:

\[
V(S_T) = \begin{cases} 
S_T - K, & \text{if } S_T > K \\
0, & \text{otherwise}
\end{cases}
\]

(b) Estimate the price of the put option by generating \( N = 10,000 \) samples of the stock price \( S_T \) at maturity time \( T = 100 \). Use the parameter values \( r = 0.05 \), \( \sigma = 0.20 \), \( s_0 = 10 \), \( K = 10 \). The payoff of the put option is given by:

\[
V(S_T) = \begin{cases} 
K - S_T, & \text{if } S_T < K \\
0, & \text{otherwise}
\end{cases}
\]
(c) Estimate the price of an exotic option by generating $N = 10,000$ samples of the stock price $S_T$ at maturity time $T = 100$. Use the parameter values $r = 0.05$, $\sigma = 0.20$, $s_0 = 10$, $K = 10$. The payoff of the exotic option is given by:

$$V(S_T) = \begin{cases} 
(S_T - K)^2, & \text{if } S_T > K \\
0, & \text{otherwise}
\end{cases}$$

2) In this problem you will use histogram methods to empirically study the behavior of the probability distribution of the error of a Monte-Carlo estimate. For a large number of samples we expect (by the Central Limit Theorem) that the error will take the form of a normal distribution.

Suppose we use the Monte-Carlo method to estimate the integral:

$$I = \int_{0}^{1} F(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} F(X_i)$$

where $X_i$ is the $i^{th}$ sample of the uniform random variable on $[0, 1]$.

Now, for any given run of the Monte-Carlo estimate the error of the approximation will be a random quantity. We define the random variable for the error as:

$$Z = \frac{1}{N} \sum_{i=1}^{N} F(X_i) - \int_{0}^{1} F(x)dx.$$

Now to study the errors that arise, run the Monte-Carlo estimation procedure $M = 10,000$ separate times to obtain samples of the error $Z_i$. Compute the probability density $\rho_Z(x)$ of the error using the command `hist()` as follows:

```matlab
[N,x] = hist(samplesZ,100);
deltaX = x(2) - x(1);
rho_Z = N/(sum(N)*deltaX);
```

Compare the histogram estimate with the normal distribution with mean 0 and variance $\sigma^2/N$:

```matlab
figure(1);
clf;
bar(x,rho_Z);
hold on;
plot(x,rho_analytic(x),'r-')
xlabel('x');
ylabel('$\rho(x)$);
```

where

$$\sigma^2 = \text{var } [F(X)].$$

If these probability densities match well, then the theory does a good job of predicting the "typical size" of the error in a Monte-Carlo estimate as $\frac{\sigma}{\sqrt{N}}$ (one standard deviation). For the problems below you will perform the above procedure to see how well the theory describes
the error as we increase the number of samples $N$ used.

(a) For the Monte-Carlo method applied to estimate the integral

$$I = \int_0^1 x^2 \, dx$$

give a plot of the above histogram estimate of the probability density of the error $\rho_z(x)$ and compare it to the plot of the normal distribution, when using first $N = 10$ samples, then $N = 100$, then $N = 10,000$ samples. In each case, how well does the theoretical error (normal distribution) agree with your numerical estimate of the error distribution (histogram estimate)? By what factor do you expect the error to decrease if we double $N$?

(b) For the Monte-Carlo method applied to estimate the integral

$$I = \int_0^1 3 \, dx.$$

give a plot of the above histogram estimate of the probability density of the error $\rho_z(x)$ and compare it to the plot of the normal distribution, when using first $N = 10$ samples, then $N = 100$, then $N = 10,000$ samples. In each case, how well does the theoretical error (normal distribution) agree with your numerical estimate of the error distribution (histogram estimate)? By what factor do you expect the error to decrease if we double $N$?

(c) For the Monte-Carlo method applied to estimate the integral

$$I = \int_0^1 e^x \, dx$$

give a plot of the above histogram estimate of the probability density of the error $\rho_z(x)$ and compare it to the plot of the normal distribution. How well does the theoretical error (normal distribution) agree with your numerical findings (histogram estimate)?