Name: Solution key

Midterm Exam:
Professor: Paul J. Atzberger
Numerical Analysis II, 104B
February 13th, 2007

Scoring:
Problem1: ____________

Problem2: ____________

Problem3: ____________

Problem4: ____________

Directions: Answer each question carefully and be sure to show all of your work. If you have any questions please feel free to ask.
Problem 1: Use the specified quadrature rule to estimate the following integrals using the indicated number of nodal points $n$. Compute using 4 digits of precision. In each problem compute the exact value of the integral and give the relative error of the estimate.

a) Use the Composite Trapezoidal Rule to estimate $\int_0^2 xe^{-x^2}dx$ with $n = 4$. What is the exact value of this integral? What is the relative error of the estimate?

<table>
<thead>
<tr>
<th>$X_k$</th>
<th>$f(X_k)$</th>
<th>$h/2 (f(X_k) + f(X_{k+1}))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$0.3694$</td>
<td>$0.1893$</td>
</tr>
<tr>
<td>2</td>
<td>$0.1839$</td>
<td>$0.1315$</td>
</tr>
<tr>
<td>3</td>
<td>$0.0527$</td>
<td>$0.0486$</td>
</tr>
<tr>
<td>4</td>
<td>$0.1092$</td>
<td></td>
</tr>
</tbody>
</table>

$h = \frac{1}{2}$

$I = \int_0^2 xe^{-x^2}dx \approx \sum_{k=0}^{3} \frac{h}{2} (f(X_k) + f(X_{k+1})) = 0.4668 \approx \frac{\pi}{2}$

$I = \left. -\frac{1}{2} e^{-x^2} \right|_{x=0}^{x=2} = \frac{1}{2} (1 - e^{-4}) = 0.4908$

$relative\ error = \frac{|I - I|}{|I|} = 0.04804$

b) Use the Composite Simpson’s Rule to estimate $\int_0^1 \frac{1}{x} dx$ with $n = 4$. What is the exact value of this integral? What is the relative error of the estimate?

<table>
<thead>
<tr>
<th>$X_k$</th>
<th>$f(X_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{6} = 0.1667$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{7} = 0.1429$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{8} = 0.1250$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{9} = 0.1111$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{10} = 0.1$</td>
</tr>
</tbody>
</table>

$h = 1$

$I = \int_0^1 \frac{1}{x} dx \approx \frac{h}{3} \left[ f(0) + f(1) + f(1) + \frac{h}{3} \left[ f(6) + 4f(7) + f(8) \right] + \frac{h}{3} \left[ f(8) + 4f(9) + f(10) \right] \right]$

$I = 0.4198 \approx \frac{\pi}{2}$, $I = \ln(10) - \ln(6) = 0.5108$

$relative\ error = \frac{|I - I|}{|I|} = 0.0508$
c) Compute the exact value of \( I \) and give the absolute errors for each of the above estimates.

\[
I = \int_0^1 x^\frac{2}{3} \, dx = \left. \frac{1}{\frac{2}{3}} x^{\frac{2}{3}} \right|_{x=0}^{x=1} = \frac{3}{2} \approx 1.5
\]

\[E_1 = |I_1 - I| = \frac{|0.774 - 1.5|}{0.774} \approx 0.2334\]

\[E_2 = |I_2 - I| = 0.2334\]

d) Compute the mean and variance associated with each of the estimates:

\[\mu = \int_0^1 \left( \frac{x^2}{\rho(x)} \right) \rho(x) \, dx\]

\[\sigma^2 = \int_0^1 \left( \frac{x^2}{\rho(x)} - \mu \right)^2 \rho(x) \, dx\]

\[M_1 = \frac{1}{3}\]

\[E_1^{(2)} = \int_0^1 \left( x^\frac{2}{3} - \frac{1}{3} \right)^2 \, dx = \int_0^1 x^4 - \frac{2}{3} x^\frac{5}{3} + \frac{1}{9} \, dx = \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{1}{45}\]

\[M_2 = \frac{1}{3}\]

\[E_2^{(2)} = \int_0^1 \left( \frac{x}{3} - \frac{1}{3} \right)^2 \frac{2}{3} x \, dx = \int_0^1 \frac{x^3}{2} - \frac{2}{3} x^\frac{5}{3} + \frac{1}{9} \, dx = \frac{1}{8} - \frac{2}{9} + \frac{1}{9} = \frac{1}{72}\]

e) For \( N = 10 \) samples give the theoretical error (standard deviation \( \sigma \) of the error) for each of the Monte-Carlo estimates. Which estimate performs better for this set of random samples? which is predicted to perform better in theory as \( N \) becomes large?

\[E_1 = \frac{1}{\sqrt{1450}}\]

\[E_{N_1} \sim \frac{E_1}{\sqrt{N}} = \frac{1}{\sqrt{1450}} = 0.0943\]

\[E_2 = \frac{1}{\sqrt{22}}\]

\[E_{N_2} \sim \frac{E_2}{\sqrt{N}} = \frac{1}{\sqrt{220}} = 0.0323\]

estimate 2 performs better for both these samples and is expected theoretically to perform better for large \( N \).
**Problem 2:** Monte-Carlo Method: Compute the Monte-Carlo estimate of \( I \) using random samples of the random variable \( U \), which has uniform distribution \( \rho(x) = 1 \) on \([0, 1] \), and of the random variable \( V \), which has distribution \( \rho(x) = 2x \) on \([0, 1] \). Use the following random samples:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( U_k )</th>
<th>( V_k )</th>
<th>( U_k^2 )</th>
<th>( V_k^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9501</td>
<td>0.9747</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2311</td>
<td>0.4808</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.6068</td>
<td>0.7790</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.4860</td>
<td>0.6971</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.8913</td>
<td>0.9441</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.7621</td>
<td>0.8730</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.4565</td>
<td>0.6756</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.0185</td>
<td>0.1360</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.8214</td>
<td>0.9063</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.4447</td>
<td>0.6669</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{sum} \ G_1 = 4.4585 \text{ G} \quad 4.0169 \text{ G} \quad 5.6885 \]

(a) Give the Monte-Carlo estimate of

\[ I = \int_0^1 x^3 \, dx \]

when using the uniform random variable \( U \), \( x_n = U_n \)

\[ I \approx \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \frac{1}{10} \left( 4.0169 \right) = 0.40169 \approx I_1 \]

(b) Give the Monte-Carlo estimate of \( I \) when using the non-uniform random variable \( V \), \( x_n = V_n \)

\[ I \approx \frac{1}{N} \sum_{n=1}^{N} \frac{f(x_n)}{\rho(x_n)} = \frac{1}{N} \sum_{n=1}^{N} \frac{x_n^3}{2x_n} = \frac{1}{2N} \sum_{n=1}^{N} x_n^2 \]

\[ = \frac{1}{2-0} (7.1335) = 0.3567 \approx I_2 \]
Problem 3: Euler’s Method: Compute approximate solutions to the following ODE's using Euler’s Method with the indicated time step $h$ and indicated number of time steps $N$.

a) Compute using Euler’s method an approximation to the ODE:

$$\begin{align*}
\frac{dy}{dt} &= -2y^2 + t \\
y(0) &= 1
\end{align*}$$

with $h = 0.25$ and $N = 2$.

$$\begin{align*}
y_{n+1} &= y_n + f(y_n, b_n) \cdot h \\
y_0 &= 1 \\
y_1 &= \frac{1}{2} (-2 \cdot 1 + 0) \cdot 0.25 = 0.5 \\
y_2 &= \left( -2 \cdot (-0.5)^2 + 0.25 \right) \cdot 0.25 + 0.5 \\
    &= 0.4375
\end{align*}$$

b) Compute using Euler’s method an approximation to the ODE: (5 digits precision)

$$\begin{align*}
\frac{dy}{dt} &= -y \\
y(0) &= 1
\end{align*}$$

over a single time step, but with different values of $h = 0.1$, $h = 0.05$, $h = 0.0025$, $h = 0.00125$.

$$\begin{align*}
y_{n+1} &= y_n + f(y_n, b_n) \cdot h \\
y_0 &= 1 \\
y(0, 0.1) &= 1 + (-1) \cdot 0.1 = 0.9 \\
y(0, 0.05) &= 1 + (-1) \cdot 0.05 = 0.95 \\
y(0, 0.0025) &= 1 + (-1) \cdot (0, 0.0025) = 0.9975 \\
y(0, 0.00125) &= 1 + (-1) \cdot (0, 0.00125) = 0.99875
\end{align*}$$

$$\begin{align*}
y(h) &= e^{-h} \\
y(0.1) &= 0.90483 \\
y(0.05) &= 0.95123 \\
y(0.0025) &= 0.99753 \\
y(0.00125) &= 0.998758
\end{align*}$$
c) For part (b) give the absolute error of the approximate Forward-Euler solution \( \tilde{y}(h) \) with respect to the exact solution \( y(h) \) for each value of \( h \). Give these errors in a table along with a plot of the errors vs. \( h \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \tilde{y}(h) )</th>
<th>( y(h) )</th>
<th>( \varepsilon(h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.99</td>
<td>0.90483</td>
<td>0.00483</td>
</tr>
<tr>
<td>0.05</td>
<td>0.95</td>
<td>0.95136</td>
<td>0.00136</td>
</tr>
<tr>
<td>0.025</td>
<td>0.975</td>
<td>0.97532</td>
<td>0.00031</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.9875</td>
<td>0.98758</td>
<td>0.00008</td>
</tr>
</tbody>
</table>

\[ \varepsilon(0.1) \]