Math 34A
Week 2
13 October 2017

Math 34A — Week 2

Lines A line is a relationship between two variables, commonly referred to as x and y, that has a constant slope (meaning the same slope everywhere). The two most common ways we write this relationship are

$$y = mx + b, \qquad m = \text{ slope}, b = y\text{-intercept}$$

and

$$y - y_0 = m(x - x_0), \qquad \text{for some } (x_0, y_0) \text{ on the line.}$$

These equations are describing the same relationship between x and y.[1] Since these equations are actually describing the same thing, which of them we decide to use is really just a matter of which we think will be easier in the current situation.

One thing we can do with lines is make linear interpolations and extrapolations. If we have two points and we think the relationship between the variables involved is linear, we can find points we don't know by looking at the line formed by the two points. If we look between the points, we are interpolating. If we look outside the two points, we are extrapolating.

[a] Actually, if we write the equation of a line with the second one using the y-intercept as our (x_0, y_0) (so, $(x_0, y_0) = (0, b)$), we get

$$y - b = m(x - 0)$$

or

$$y = mx + b.$$

Line problems

3. Find the equation of the line through $(2, a)$ and $(5, b)$ in both forms.

11. Find where the line which passes through the two points $(1, 2)$ and $(3, 5)$ intersects the line through $(2, 1)$ and $(5, -6)$.

13. Car A leaves Santa Barbara at noon traveling south toward Los Angeles at 55 mph. At 1pm car B leaves Los Angeles traveling north toward Santa Barbara at 90 mph. The distance between the cars at noon was 100 miles. If car B does not get pulled over, how many minutes after car B leaves will they pass each other?

14. Two car companies Elvis and Hearts rent cars. The Elvis car costs 30 dollars plus 20 cents per mile. The Hearts car costs 50 dollars with 50 free miles and then costs 15 cents per mile. On scratch paper graph the cost against distance driven for both companies. It is cheaper to rent from Elvis if you are driving less than how many miles?

17. Find the equation of the line $x = 4t + 5$ and $y = 35t$ in the form $y = mx + b$.

18. (a) What is the slope of the line $x = 9t + 6$ and $y = 4t + 2$?
    (b) Where does the line cross the x-axis?
    (c) Where does the line cross the y-axis?

21. Suppose the points $(3, 5)$ and $(17, 12)$ are on line L.
    (a) What point is midway between these points?
    (b) What points is 1/3 of the way from $(3, 5)$ toward $(17, 12)$?
    (c) If $(4, y)$ is on L, find y.
    (d) If $(x, 8)$ is on L, find x.

35. For the function $f(x) = \sqrt{x}$, we know $f(4) = 2$ and $f(9) = 3$.
    (a) Use linear interpolation to find $\sqrt{5}$.
    (b) Find the error of this calculation to 3 decimal places.
    (c) What is the percentage error?

Proportional Relationships Another way that two quantities can be said to be related is by saying that they are proportional to each other. To say that two variables, x and y, are proportional means that there is some number k, which we call the proportionality constant, for which

$$y = kx.$$

This is again descrbining a type of relationship between x and y.[2]

Sometimes, we have y proportional to some formula of x. For example, we could say y is proportional to x^3 and write

$$y = kx^3.$$

Another common relationship is inverse proportionality. We say that x and y are inversely proportional if

$$y = k \cdot \frac{1}{x}$$

for some number k.

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[b] Notice that this relationship is actually a sort of line with y-intercept 0!

Proportionality problems

26. The rate R that a certain disease spreads is proportional to the number of infected individuals and is also proportional to the number of uninfected individuals. The total population is P and the number of infected individuals is D. Express the rate that the disease spreads in terms of this information. Use C as your proportionality constant.

28. The weight of a sphere is proportional to its radius cubed. If a sphere of diameter 1 cm has a mass of 8 grams, what diameter sphere has a mass of 216 grams?

30. The cost of flying a passenger plane consists of fuel cost and hourly pay for the flight crew. The faster a plane flies, the more fuel it uses to fly each mile. Suppose that the cost of fuel to fly one mile is proportional to the speed of the plane. The plane flies a distance of D miles.

    (a) Express the time T taken to fly D miles in terms of the velocity V of the plane.

    (b) Express the cost of the fuel F in terms of the velocity V and the distance D. Use K as your proportionality constant.

    (c) Express the cost of the flight crew f in terms of the velocity V, the distance D, and the hourly pay P.

    (d) Express the total operating costs C in terms of the velocity V, the distance D, and the hourly pay P.

    (e) If the flight crew costs 200 dollars per hour and the fuel costs 100 dollars per hour when the speed is 300mph, how much would it cost to fly 900 miles at 450 mph?

Exponential Relationships Another type of relationship between x and y is an exponential one. We say that y grows exponentially in x if there is some number a so that

$$y = a^x.$$

Note this is different from $y = x^a$. Of course, exponents satisfy some properties that we are used to:

$$a^n \cdot a^m = a^{n+m}$$
$$\frac{a^n}{a^m} = a^{n-m}$$
$$(a^m)^n = a^{m \cdot n}$$
$$(ab)^n = a^n \cdot b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$a^{-n} = \frac{1}{a^n}, a \neq 0$$
$$a^0 = 1, a \neq 0$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Exponent problems

45. The half-life of element X is 60 years. If there are 32 g initially,

    (a) How much is there after 240 years? (no exponents needed)

    (b) How much is there after 17 years?

47. (a) The population of rabbits on an island is growing exponentially. In 1910 there were 200 rabbits and by 1920 this population had grown to 1400 rabbits. Give an equation for the population of rabbits t years after 1910.

    (b) Instead, we can write an equation in terms of the doubling time of the rabbits. An equation like this would look like

    $$R(t) = R \cdot 2^{t/K}.$$

    Find R and K for this equation.

    (c) What is the doubling time for this population of rabbits?

56. On the planet Maximillian live Sprogs and Graks. Initially, there were 2400 sprogs and 300 Graks. The population of Sprogs doubles every 18 years and that of Graks doubles every 9 years. How many Graks were there after $4\frac{1}{2}$ years?

57. In the year 1900, in the country Acirema, there were 100 Lawyers and 4 million people. Every 10 years, the number of Lawyers doubles, and the population increases by 2 million. Let t be the number of years after 1900. Thus $t = 3$ corresponds to 1903. Find the equation involving t whose solution tells you in which year 20 percent of the population are Lawyers.

58. E. Coli bacteria are growing in a hamburger exponentially. Initially there are 100,000 bacteria. After 20 minutes there are 150,000. How many are there after an hour? Find this without resorting to logarithms.

Logarithms Logarithms are the opposite of exponents.[3] We can think about them like this:

$$b^p = y \qquad \Longrightarrow \qquad \log_b y = p.$$

Logarithms will be very helpful to us because they have some helpful properties that let us solve exponential equations! Some common ones are:

$$\log_b (b) = 1$$
$$\log_b (x^p) = p \log_b x$$

$$\log_b (xy) = \log_b x + \log_b y$$
$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

[c]Actually, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$.

Logarithm problems

45. The half-life of element X is 60 years. If there are 32 g initially,

    (a) When will 16 g remain? (no logs needed)

    (b) When will 26 g remain?

52. Express 4^y as a power of 10. In other words, find x so that $4^y = 10^x$.

56. On the planet Maximillian live Sprogs and Graks. Initially, there were 2400 sprogs and 300 Graks. The population of Sprogs doubles every 18 years and that of Graks doubles every 9 years. When are there as many Sprogs as Graks?

58. E. Coli bacteria are growing in a hamburger exponentially. Initially there are 100,000 bacteria. After 20 minutes there are 150,000. How many are there after an hour? Find this by coming up with an equation for the number of bacteria and solving for $t = 1$.