QUIZ IS DUE AT THE BEGINNING OF SECTION ON THURSDAY, MAY 31
This quiz is open notes and book, but NO COLLABORATION.

Math 6A — Quiz 4

Name:_____ Section Time:_____

Complete the following problems, making sure to SHOW ALL WORK. If you're stuck on something, CLEARLY EXPLAINING what you do know will get you partial credit!

1. Consider the curve parametrized by

$$\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \theta \rangle, \qquad \theta \in [0, 2\pi].$$

(a) Compute the arc length of this curve.

(b) Reparametrize the curve with respect to arclength. Be sure to give the bounds for the new parameter as well.

(c) Calculate the integral of the function $f(x, y, z) = xyz$ along this curve.

(d) Calculate the integral of the vector field ∇f along this curve, with f being the function given in part (c).

(a) We compute the arc length as

$$\begin{aligned} s = \int_C \|r'(\theta)\| \, d\theta &= \int_0^{2\pi} \|\langle -sin(\theta), \cos(\theta), 1\rangle\| \, d\theta \\ &= \int_0^{2\pi} \sqrt{\sin^2(\theta) + \cos^2(\theta) + 1^2} \, d\theta \\ &= \int_0^{2\pi} \sqrt{2} \, d\theta \\ &= 2\sqrt{2}\pi - 0 \\ &= 2\sqrt{2}\pi. \end{aligned}$$

(b) We begin similarly as before, but integrating from angle 0 to angle θ:

$$\begin{aligned} s(t) &= \int_0^{\theta} \|\langle -sin(\phi), \cos(\phi), 1\rangle\| \, d\phi \\ &= \int_0^{\theta} \sqrt{2} \, d\phi \\ &= \sqrt{2}\theta. \end{aligned}$$

Now, we solve for θ to find

$$\theta = \frac{s}{\sqrt{2}},$$

and

$$r(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle.$$

Where before we had $0 \leq \theta \leq 2\pi$, we now have $\sqrt{2}(0) \leq s \leq \sqrt{2}(2\pi)$, or $s \in [0, 2\sqrt{2}\pi]$. Notice that this is unsurprising since we have parametrised with respect to arc length. We expect to begin at an arc length of 0 and finish traversing the curve when s is the total length of the entire curve.

(c) We now compute

$$\begin{aligned}
\int_C f\, ds &= \int_0^{2\pi} f(r(\theta)) \left\| r'(\theta) \right\| \, d\theta \\
&= \int_0^{2\pi} \theta \sin(\theta) \cos(\theta) \sqrt{2}\, d\theta \\
&= \sqrt{2} \int_0^{2\pi} \theta \left(\frac{1}{2} \sin(2\theta) \right) \, d\theta && (\sin(2\theta) \text{ identity}) \\
&= \frac{1}{2\sqrt{2}} \int_0^{2\pi} 2\theta \sin(2\theta)\, d\theta \\
&= \frac{1}{2\sqrt{2}} \left(-\theta \cos(2\theta) \Big|_0^{2\pi} - \int_0^{2\pi} -\cos(2\theta)\, d\theta \right) && (\text{IBP}) \\
&= \frac{1}{2\sqrt{2}} \left(-2\pi + \int_0^{2\pi} \cos(2\theta)\, d\theta \right) \\
&= \frac{1}{2\sqrt{2}} \left(-2\pi + \frac{1}{2} \sin(2\theta) \Big|_0^{2\pi} \right) \\
&= \frac{1}{2\sqrt{2}} (-2\pi + 0) \\
&= -\frac{\pi}{\sqrt{2}}.
\end{aligned}$$

(d) We now utilise the Fundamental Theorem of Calculus to find

$$\begin{aligned}
\int_C \nabla f \cdot \, ds &= f(r(2\pi)) - f(r(0)) \\
&= f(1, 0, 2\pi) - f(1, 0, 0) \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

$\square$