The Row Reduction Algorithm

The Punch Line: Given any linear system of equations, there is a procedure which finds a particularly simple equivalent system.

Warm-Up: For each of these matrices, determine if it is in Echelon Form, Reduced Echelon Form, or neither.

(a)
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the Algorithm: Five steps transform any matrix into a row-equivalent Reduced Echelon Form matrix:

1) Identify the pivot column. This will be the leftmost column with a nonzero entry.

2) Select a nonzero entry in that column to be the pivot for that column. If necessary, interchange rows to put it at the top of the matrix.

3) Eliminate all of the nonzero entries in the pivot column by using row replacement operations.

4) Repeat steps 1)-3) on all rows you haven't yet used.

5) Eliminate all nonzero entries above each pivot, and scale each nonzero row so its pivot is 1.

1. Transform each of these matrices into Reduced Echelon Form with the above procedure:

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Interpreting the Results: The Reduced Echelon Form of the augmented matrix of a linear system can be used to find all solutions (the *solution set*) of the system at once. To do this, we write out the system corresponding to the Reduced Echelon Form matrix, then solve for all of the variables in pivot positions (we can do this easily because each one only appears in a single equation). Any remaining variables are called *free variables*, and can take on any value in a solution.

2. Find the solution set of each of these linear systems:

(a)

$$\begin{aligned} 2x_1 + 4x_3 &= 0 \\ x_2 + x_3 &= 3 \\ x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} x + y &= 1 \\ 4y - 2x &= -2 \\ -x &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

Under the Hood: The Reduced Echelon Forms of any two equivalent systems are the same. Since equivalent systems have the same solution set (by definition!), it is in some sense the simplest system with that solution set. Thus, the Row Reduction Algorithm is a way to find the simplest description of the solution set of a linear system—that works every time!