Linear Independence

The Punch Line: *Linear independence* is a property describing a collection of vectors whose span is "as big as it can be"—no vector is redundant, but each one contributes a "different direction" to the span.

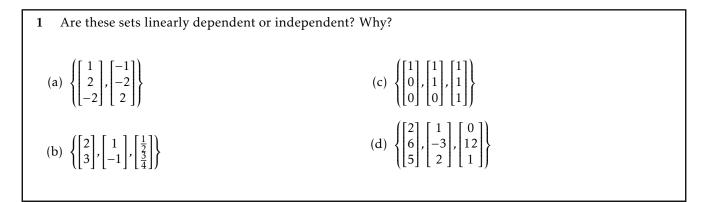
Warm-Up: Are each of these situations possible?

- (a) The vectors $\{\vec{u}, \vec{v}\}$ in \mathbb{R}^2 span all of \mathbb{R}^2 .
- (b) The vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span all of \mathbb{R}^3 .
- (c) The vectors $\{\vec{x}, -\vec{x}\}$ span all of \mathbb{R}^2 .
- (d) The vectors $\{\vec{u}, \vec{v}\}$ span all of \mathbb{R}^3 .
- (e) The span of $\{\vec{u_1}, \vec{u_2}\}$ and the span of $\{\vec{v_1}, \vec{v_2}\}$ in \mathbb{R}^3 intersect only at $\vec{0}$.
- (f) The span of $\{\vec{u_1}, \vec{u_2}\}$ and the span of $\{\vec{v_1}, \vec{v_2}\}$ in \mathbb{R}^3 are both planes and intersect only at $\vec{0}$.
- (a) Yes, two vectors can span a plane.
- (b) Yes, four vectors can span 3-space.
- (c) No, they are both on the same line, so their span is that line.
- (d) No, two vectors can span at most a plane.
- (e) Yes, but only if at least one of the spans is just a line and not a whole plane.
- (f) No, any two planes in \mathbb{R}^3 that include $\vec{0}$ (or any specific point in common) must intersect on at least a line.

Tests for Linear Independence: That the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is linearly independent means that if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, the only possibility is that $c_1 = c_2 = \cdots = c_n = 0$. That is, the only solution to the homogeneous matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

is the trivial solution: the zero vector. If there's only one vector in the set, it is linearly independent unless it happens to be $\vec{0}$. If there's only two vectors, they are linearly independent unless one is a multiple of the other (including $\vec{0}$, which is 0 times any vector). If a subset of the vectors is linearly dependent, the whole set is. Finally, a set of vectors is linearly dependent if (and only if) at least one vector is in the span of the others.



- (a) No, the second vector is -1 times the first.
- (b) No, the third vector is $\frac{1}{4}$ times the first, plus there are three vectors in \mathbb{R}^2 .
- (c) Yes, and we can check by observing there are no free variables in the REF of the coefficient matrix of the homogeneous equation described above.
- (d) No, the third vector is the sum of the first and -2 times the second.

- 2 Are each of these situations possible?
 - (a) You have a set of vectors that spans \mathbb{R}^3 . You remove two of them, and the set of vectors left behind is linearly independent.
 - (b) You have two sets of vectors in \mathbb{R}^6 . One has four vectors, and one has two vectors, and both sets are linearly independent. When you put both sets together, the resulting set of six vectors is linearly dependent.
 - (c) You have a set of three vectors which span \mathbb{R}^3 , but it is linearly dependent.
 - (d) You have a linearly dependent set of three vectors in \mathbb{R}^2 . If you remove any one of them, the other pair do not span \mathbb{R}^2 .
- (a) Yes, for example, if we started with $\left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1 \end{bmatrix} \right\}$ and removed the last two.
- (b) Yes, for example, if each vector in the set of two was a multiple of one from the set of four.
- (c) No, because if the set is linearly dependent, one vector would be in the span of the other two, and the span of two vectors is a plane, while \mathbb{R}^3 is a 3-space.
- (d) Yes, for example if the three vectors are all multiples of each other, so their span is a line.