Math 4A
Worksheet 1.7

Linear Independence

The Punch Line: *Linear independence* is a property describing a collection of vectors whose span is "as big as it can be"—no vector is redundant, but each one contributes a "different direction" to the span.

Warm-Up: Are each of these situations possible?

(a) The vectors $\{\vec{u}, \vec{v}\}$ in $\mathbb{R}^2$ span all of $\mathbb{R}^2$.

(b) The vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span all of $\mathbb{R}^3$.

(c) The vectors $\{\vec{x}, -\vec{x}\}$ span all of $\mathbb{R}^2$.

(d) The vectors $\{\vec{u}, \vec{v}\}$ span all of $\mathbb{R}^3$.

(e) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ intersect only at $\vec{0}$.

(f) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ are both planes and intersect only at $\vec{0}$.

Tests for Linear Independence: That the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent means that if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, the only possibility is that $c_1 = c_2 = \cdots = c_n = 0$. That is, the only solution to the homogeneous matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

is the trivial solution: the zero vector. If there's only one vector in the set, it is linearly independent unless it happens to be $\vec{0}$. If there's only two vectors, they are linearly independent unless one is a multiple of the other (including $\vec{0}$, which is 0 times any vector). If a subset of the vectors is linearly dependent, the whole set is. Finally, a set of vectors is linearly dependent if (and only if) at least one vector is in the span of the others.

1 Are these sets linearly dependent or independent? Why?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix} \right\}$

2 Are each of these situations possible?

(a) You have a set of vectors that spans $\mathbb{R}^3$. You remove two of them, and the set of vectors left behind is linearly independent.

(b) You have two sets of vectors in $\mathbb{R}^6$. One has four vectors, and one has two vectors, and both sets are linearly independent. When you put both sets together, the resulting set of six vectors is linearly dependent.

(c) You have a set of three vectors which span $\mathbb{R}^3$, but it is linearly dependent.

(d) You have a linearly dependent set of three vectors in $\mathbb{R}^2$. If you remove any one of them, the other pair do not span $\mathbb{R}^2$.