Linear Transformations

The Punch Line: Matrix multiplication defines a special kind of function, known as a *linear transformation*.

Warm-Up: What do each of these situations mean (geometrically, algebraically, in an application, and/or otherwise)?

(a) The product of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) The equation $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution.

(d) The set of vectors $\vec{x}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a plane in $\mathbb{R}^3$.

(e) The set of vectors $\vec{b}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a line in $\mathbb{R}^2$.

(f) For two particular vectors $\vec{x}$ and $\vec{b}$, and a matrix A, the matrix equation $A\vec{x} = \vec{b}$ is satisfied.

What They Are: A *linear transformation* is a mapping T that obeys two rules:

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in its domain,

(b) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and $\vec{u}$ in its domain.

These rules lead to the rule $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for c, d scalars and $\vec{u}, \vec{v}$ in the domain of T, and in fact $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$. That is, the transformation of a linear combination of vectors is a linear combination of the transformations of the vectors (with the same coefficients).

1 Are each of these operations linear transformations? Why or why not?

(a) $T(\vec{x}) = 4\vec{x}$

(b) $T(\vec{x}) = A\vec{x}$ for some matrix A (with the right number of columns)

(c) $T(\vec{x}) = \vec{0}$

(d) $T(\vec{x}) = \vec{b}$ for some nonzero $\vec{b}$

(e) $T(\vec{x}) = \vec{x} + \vec{b}$ for some nonzero $\vec{b}$

(f) $T(\vec{x})$ takes a vector in $\mathbb{R}^2$ and rotates it by 45° ($\frac{\pi}{4}$ radians) counter-clockwise in the plane

What They Do: Linear transformations convert between two different spaces, such as $\mathbb{R}^n$ and $\mathbb{R}^m$. If $n = m$, then we can also think of them moving around the vectors inside $\mathbb{R}^n$ (e.g., by rotation or stretching).

2 What do the linear transformations corresponding to multiplication by these matrices do, geometrically? (Try applying the matrix to a vector composed of variables, then examining the result, or multiplying by a few simple vectors and sketching what happens.)

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$