Math 4A
Key 1.9

The Matrix of a Linear Transformation

The Punch Line: Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are *all* equivalent to matrix transformations, even when they are described in other ways.

Warm-Up: What does the linear transformation corresponding to multiplication by each of these matrices do geometrically (don't worry too much about the exact values for things like rotation or scaling)?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(a) This is a reflection about the x-axis—the y coordinate of each point is negated, so vectors above the axis are moved an equal distance below it, without changing in x.

(b) This is both a rotation by 45° ($\frac{\pi}{4}$ radians), and a scaling by $\sqrt{2}$.

(c) This is a projection to the xy-plane—the z coordinate collapses down to zero while the other coordinates remain unchanged.

(d) This maps a vector in $\mathbb{R}^3$ to the vector in $\mathbb{R}^2$ that looks like its projection in the xy plane. While this is a very similar transformation to the previous one, it's important to note that this time, the result is in a different space (honest-to-goodness $\mathbb{R}^2$, rather than a plane in $\mathbb{R}^3$).

Getting the Matrix: We can write down a matrix that accomplishes any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by writing down what the transformation does to the vectors corresponding to each component (these have a single 1 and the rest of their entries as zeros, and make up the columns of the $n \times n$ *identity matrix*, which has ones down the diagonal and zeros elsewhere).

1 Write down a matrix for each of these linear transformations.

(a) In $\mathbb{R}^2$, rotation by 180° (π radians) counterclockwise.

(b) In $\mathbb{R}^3$, rotation by 180° (π radians) counterclockwise in the xz plane.

(c) In $\mathbb{R}^2$, stretching the x direction by a factor of 2 then reflecting about the line $y = x$.

(d) In $\mathbb{R}^3$, the transformation that looks like a "vertical" (that is, the z direction is the one which moves) shear in both the xz and yz planes, each with a "shear factor" (the amount the corner of the unit square moves) of 2.

[Note: Don't worry too much if this one's harder than the rest, shear transformations are hard to describe. If you get stuck, it might be a good idea to work on Problem 2 rather than sink in too much time here.]

(a) We see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is sent to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is sent to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Putting these together, we get the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) Here we get $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) Here $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

(d) Here $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (ask me if you want to talk about why).

One to One and Onto: When describing a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, we say T is *one to one* if each vector in $\mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$ (it can fail to be the image of any vector, it just can't be the image of two different ones). We say T is *onto* if each vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$ (it can be the image of more than one).

We can test these conditions with ideas we already know: T is one-to-one if and only if the columns of its matrix are linearly independent, and onto if and only if they span $\mathbb{R}^m$. An equivalent test for T being one-to-one is that the equation $A\vec{x} = \vec{0}$ (where A is the matrix of T) has only the trivial solution if and only if T is one-to-one. An equivalent test for onto is that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^m$.

2 Determine if the linear transformations with the following matrices are one-to-one, onto, both, or neither.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & 12 \\ 5 & 2 & 1 \end{bmatrix}$

(a) This is both, as $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ both spans $\mathbb{R}^2$ and is linearly independent.

(b) This is also both.

(c) This is one-to-one but not onto, as $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is linearly independent, but does not span $\mathbb{R}^3$ (in particular, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is not in their span).

(d) This is onto but not one-to-one, as $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ spans $\mathbb{R}^2$, but is not linearly independent (in particular, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$).

(e) This is neither, as $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (for example) is not in the span of the columns.

(f) This is also neither, as the columns are linearly dependent and do not span $\mathbb{R}^3$.

Why does the $A\vec{x} = \vec{0}$ test work? If $A\vec{x} = A\vec{y}$, then $A(\vec{x} - \vec{y}) = \vec{0}$. If x and y weren't the same to begin with, then their difference is mapped to $\vec{0}$ by A as a consequence of them having the same value for the product. Similarly, if $A\vec{z} = \vec{0}$ for a nonzero $\vec{z}$, then $A(\vec{x} + \vec{z}) = A\vec{x} + A\vec{z} = A\vec{x}$, even though $\vec{x} \neq \vec{x} + \vec{z}$.