Math 4A
Key 2.1
Lecture 10/20/17

Matrix Operations

The Punch Line: Various operations combining linear transformations can be realized with some standard matrix operations.

Addition and Scalar Multiplication: Just like with vector operations, the sum of matrices and the multiplication by a *scalar* (just a number, as opposed to a vector or matrix) are done component-by-component.

1 Try the following matrix operations:

(a) $3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(a) This gives $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$.

(b) This gives $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

(c) This gives $\begin{bmatrix} -1 & 3 \\ -5 & 4 \end{bmatrix}$.

(d) This one is $\begin{bmatrix} 3 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Matrix Multiplication: To multiply two matrices, we create a new matrix, each of whose columns is the result of the matrix-vector product of the left matrix with the corresponding column of the right matrix (the product will have the same number of rows as the left matrix, and the same number of columns as the right matrix). To get the ij entry (ith row and jth column) we could multiply the ith row of the left matrix with the jth column of the right matrix.

2 Multiply these matrices (if possible, otherwise say why it isn't):

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

(a) We can compute this as $\left[\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$

(b) We can compute the upper left $(1,1)$ entry as $\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$, the upper right $(1,2)$ as $\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$, the lower left $(2,1)$ as $\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$, and the lower right $(2,2)$ as $\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$ to get the resulting matrix $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) Using either method (I think the "linear combination of columns" is slightly easier, but your results may vary), we get this as $\begin{bmatrix} 2 & 1 & 3 \\ 2 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$.

(d) This is impossible—the left matrix has 4 columns, the right has 2 rows, and these numbers must match up for the matrix multiplication procedure to be well-defined.

Transpose: The last matrix operation for today is the *transpose*, where you switch the roles of rows and columns. That is, if you get an $n \times m$ matrix, its transpose will be $m \times n$.

3 Compute the following operations for the matrices given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) A^T

(b) B^T

(c) C^T

(d) $(BA)^T$

(e) $A^T B^T$

(f) $(BAC)^T$

(g) AA^T

(h) $A^T A$

(i) $(AA^T - B)^T$

(a) This is $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$.

(b) This is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(c) This is $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$.

(d) The product $BA = \begin{bmatrix} 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix}$, so its transpose is $\begin{bmatrix} 0 & 0 \\ 2 & -2 \\ 4 & -4 \end{bmatrix}$.

(e) The product of transposes is also $\begin{bmatrix} 0 & 0 \\ 2 & -2 \\ 4 & -4 \end{bmatrix}$.

(f) This is $C^T A^T B^T = \begin{bmatrix} 0 & 0 \\ -2 & 2 \\ 0 & 0 \end{bmatrix}$.

(g) This is $\begin{bmatrix} 14 & -2 \\ -2 & 2 \end{bmatrix}$.

(h) This is $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & 6 & 10 \end{bmatrix}$. Note that it has different dimensions than AA^T.

(i) This is $AA^T - B^T = \begin{bmatrix} 13 & -1 \\ -1 & 1 \end{bmatrix}$.

What do these operations mean? Matrix addition and scalar multiplication correspond to adding and scaling the results of applying the linear transformation of the matrix, respectively. Matrix multiplication corresponds to composing the two linear transformations (applying one to the result of another). Transposition is a little weirder, and corresponds to switching the roles of variables and coefficients in a linear equation.