Math 4A
Key 2.2
Lecture 10/23/17

The Inverse of a Matrix

The Punch Line: Undoing a linear transformation given by a matrix corresponds to a particular matrix operation known as *inverse*.

Warm-Up: Are the following vector operations reversible/invertible?

(a) $T(\vec{x}) = 4\vec{x}$

(b) $T(\vec{x})$ is counterclockwise rotation in the plane by 45° ($\frac{\pi}{4}$ radians)

(c) $T(\vec{x}) = \vec{0}$

(d) $T(\vec{x}) = \vec{x} + \vec{b}$

(e) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$

(a) Yes, with inverse $T^{-1}(\vec{x}) = \frac{1}{4}\vec{x}$.

(b) Yes, with an inverse given by clockwise 45° rotation.

(c) No, because we can't tell what the input was if everything goes to the same place.

(d) Yes, with inverse $T^{-1}(\vec{x}) = \vec{x} - \vec{b}$; note that this is not a linear transformation unless $\vec{b} = \vec{0}$.

(e) Yes, by multiplying by the same matrix again (switching components twice puts them back where they began).

(f) No, because $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ are both sent to $\vec{0}$, so you can't tell which you started with by looking at the result.

The Inverse: The *inverse* of an $n \times n$ matrix A is another matrix B that satisfies the two matrix equations $AB = I_n$ and $BA = I_n$, where the *identity matrix* I_n has ones on the diagonal and zeroes everywhere else. We use the notation A^{-1} to refer to such a B (which, if it exists, is unique).

We can find the inverse of a matrix by applying row operations to the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$ (which is augmented with the n columns of the identity matrix, rather than a single vector). If the left part of the augmented matrix can be transformed by row operations to I_n, then the right part will be transformed by those row operations to A^{-1}. If the system is inconsistent, the matrix A is not invertible (and we may call it *singular*).

1 Find the inverse of these matrices (you may want to check your results by multiplying the result with the original matrix):

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 7 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(a) We find the REF of the augmented matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, which is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, so the inverse of the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Indeed, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Compare to the argument in part e) of the Warm-Up.

(b) This is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$.

(c) This is $\begin{bmatrix} -2 & 1 \\ -7 & 3 \end{bmatrix}$.

(d) This is $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Relevance to Matrix Equations: The inverse of a matrix allows you to "reverse engineer" a matrix equation, in the sense that if $A\vec{x} = \vec{b}$ and A is invertible, then $\vec{x} = A^{-1}\vec{b}$ is a solution to the original equation. In fact, it is the unique solution to the equation!

2 Use the inverses computed previously to solve these matrix equations:

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 7 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} a \\ a+1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(a) We use the inverse calculated previously to get $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) Here we get $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$.

(c) We use the matrix as follows: $\begin{bmatrix} -2 & 1 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} a \\ a+1 \end{bmatrix} = \begin{bmatrix} 1-a \\ 3-4a \end{bmatrix}$.

(d) Similarly, $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{-a+b+c}{2} \\ \frac{a-b+c}{2} \\ \frac{a+b-c}{2} \end{bmatrix}$.

Computing the inverse of a matrix reveals the structure of how to invert the linear transformation it represents. As the book notes, it can be faster to simply perform row operations to find a solution to any particular matrix equation. However, looking at the inverse matrix can give a more geometric idea of what undoing some particular operation is—to undo a rotation and shear requiring a different shear and rotation in the opposite direction, for example.