Math 4A
Worksheet 2.3
Lecture 10/25/17

Characterizing Invertible Matrices

The Punch Line: There are many equivalent conditions to determine if a matrix is invertible, and describe properties of ones that we know are invertible.

Warm-Up: How big is the solution set of the homogeneous equation with these matrices (is it finite or infinite? what is its dimension?)? How about the span of their columns?

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

Matrix Conditions: Our first definition for invertible matrices states that A is invertible if some other matrix C makes the equations $AC = I_n$ and $CA = I_n$ simultaneously true. We can show that if A is invertible, so is A^T, that the inverse of A^T is C^T, and that if either one of the two equations in the definition is true, the other one must be as well, by playing around with transposing the equations.

1 Can each of these things happen? Do they have to be true?

(a) A is invertible and the matrix C with $AC = I_n$ is also invertible

(b) C is an inverse to both A and A^T (that is, $CA = I_n$ and $CA^T = I_n$)

(c) $CA = I_n$ and $AD = I_n$, but $C \neq D$

(d) $ABCD = I_n$, but $AB \neq I_n$ and $CD \neq I_n$

Equation Conditions: We also have conditions based on the homogeneous and inhomogeneous equations involving the matrix. We know A is invertible if it is square ($n \times n$) and its columns span $\mathbb{R}^n$ or are linearly independent (for square matrices these are equivalent, though not in general). That is, it has n pivots (so its EF's have pivots in every column, and its REF is I_n) or no free variables. That is, the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ (which will turn out to be unique) or $A\vec{x} = \vec{0}$ has *only* the trivial solution.

2 Are these matrices invertible?

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 3 & 1 & -4 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Why do the columns have to be linearly independent and span all of $\mathbb{R}^n$? If they were not linearly independent, there would be multiple solutions to $A\vec{x} = \vec{b}$, so we couldn't define $A^{-1}\vec{b} = \vec{x}$—we wouldn't know which to choose! And if they did not span $\mathbb{R}^n$, then there would be some $\vec{b}$ outside their span where we couldn't find any $\vec{x}$ so that $A\vec{x} = \vec{b}$—we'd again have a problem defining the inverse, but this time instead of having to many possible answers $\vec{x}$, we wouldn't have any!