Math 4A
Worksheet 4.2
Lecture 11/06/17

Column and Null Spaces

The Punch Line: The sets of vectors we've been most interested in so far in the course—solution sets (to homogeneous systems) and spans—are in fact subspaces!

Warm-Up: Can these situations happen?

(a) A vector $\vec{x}$ is in both the null space and column space of a 3×5 matrix

(b) A vector $\vec{x}$ is in both the null space and column space of a 2×2 matrix

(c) A vector $\vec{x}$ is in neither the null space nor column space of a 2×2 matrix

(d) A vector $\vec{x}$ is in neither the null space nor column space of an invertible 4×4 matrix

Null Spaces: The *null space* (also called the *kernel*) of a linear transformation T in the vector space V is the set of all vectors $\vec{x}$ that are mapped to $\vec{0} \in V$ by T: $T(\vec{x}) = \vec{0}$. For $\mathbb{R}^n$ and $T(\vec{x}) = A\vec{x}$ for a matrix A, we can explicitly describe the vectors in the null space by finding a parametric form for the solution set of the homogeneous equation $A\vec{x} = \vec{0}$. The vectors attached to each parameter span the null space.

1 Describe the null spaces of the following linear transformations:

(a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$

(b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \vec{x}$

(c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

(d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$

(e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix} \vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x}$

(g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*

*This is something of a challenge problem; it should help you understand null spaces, but it probably won't be on an exam.

Column Spaces and Range: The *column space* of a matrix is the span of its columns. For more general linear transformations, the analogous concept is *range*—the set of vectors in the vector space V that can be reached by applying the linear transformation. In $\mathbb{R}^n$, we can get the column space as just the span of the columns (although we can describe it more succinctly if we eliminate linearly dependent columns).

2 Describe the range of these linear transformations. What is their dimension? Try to find a spanning set with only that many vectors. See if you can relate these situations to the null spaces you found on the last page.

(a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$

(b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \vec{x}$

(c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

(d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$

(e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix} \vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x}$

(g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*

*This is again a challenge problem. What could the dimension be here?

What's going on with the linear transformation in part (d)? When (part of) the column space is in the null space, the matrix is sending vectors somewhere it will send to zero. If we applied the transformation twice (or, in general, enough times), it would send all vectors to zero. It's kind of a drawn-out process: send vectors matching some description (in some span) to zero, then change other vectors to take their places. It's important to remember that the null space is describing where vectors are *before* the transformation, while the column space is describing *after*.