Eigenvalues and Where to Find Them

The Punch Line: Finding the eigenvalues of a matrix boils down to finding the roots of a polynomial.

Warm-Up: What are the eigenvalues of these matrices? What is the dimension of each eigenspace?

[Note: you shouldn't have to do many computations here—just look at Echelon Forms and try to see how many free variables there will be.]

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(e) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) We can see that the only value of λ for which $A - \lambda I_3$ has a non-trivial null space is $\lambda = 2$. In this case, $A - 2I_3 = O$ (the matrix of all zeroes), so the null space is all of $\mathbb{R}^3$, which is three-dimensional. This means that E_2, the 2-eigenspace, has dimension three, and thus that A acts everywhere as if it were the scalar 2 (in some sense, this kind of matrix—with the same value along the diagonal and zeroes everywhere else—is the simplest possible).

(b) The eigenvalues here are $\lambda = 1$, 2, and 4. For 1 and 4, we can see $A - \lambda I_4$ will have three pivots remaining, so the eigenspaces will have dimension 1 (three pivots means one free variable here). Looking at $A - 2I_4$, we see it is

$$A - 2I_4 = \begin{bmatrix} -1 & 2 & 4 & 8 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The first, third, and fourth columns are clearly pivot columns here, so E_2 also has dimension 1, even though 2 appears twice on the diagonal.

(c) The eigenvalues are again $\lambda = 1$, 2, and 4. Again, $\dim E_0 = \dim E_4 = 1$, but this time

$$A - 2I_4 = \begin{bmatrix} -1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

which only has two pivots. Thus, $\dim E_2 = 2$ for this matrix.

(d) The eigenvalues are $\lambda = a$, b, and c. Each will have a one-dimensional eigenspace, unless two or all three are the same, in which case that eigenvalue will have a two- or three-dimensional eigenspace.

(e) The only eigenvalue is $\lambda = a$. If $b = 0$, then E_a has dimension 2, otherwise it has dimension 1.

(f) The only eigenvalue here is $\lambda = 0$. There are three pivots, so E_0 has $5 - 3 = 2$ dimensions.

The Characteristic Equation: If λ is an eigenvalue of the matrix A, that means there is some nonzero $\vec{v} \in \mathbb{R}^n$ that satisfies the equation $A\vec{v} = \lambda\vec{v}$. Then $(A - \lambda I_n)\vec{v} = \vec{0}$ (from putting all terms with $\vec{v}$ on the same side), so $(A - \lambda I_n)$ is a non-invertible matrix (it has nontrivial null space, because $\vec{v} \neq \vec{0}$). Since we know that a matrix being not invertible is equivalent to its determinant being zero, we can check when the equation $\det(A - \lambda I_n) = 0$ is true. This gives a polynomial equation in λ of degree n (why?), so if we can find the roots of the polynomial, we know all of the eigenvalues. This equation is known as the *characteristic equation*.

1 What is the characteristic equation for each of these matrices?
[Note: You need not solve the characteristic equation yet.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

(a) We look at $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$. This has determinant $(1-\lambda)(2-\lambda)$, so our characteristic equation is $(1-\lambda)(2-\lambda) = 0$. This is already factored, so we can confirm it gives eigenvalues 1 and 2.

(b) Here we have $A - \lambda I_2 = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$, so the characteristic equation is $(-\lambda)(-\lambda) - (1)(1) = \lambda^2 - 1 = 0$.

(c) Here we similarly have $A - \lambda I_3 = \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$. Taking the determinant and setting it to zero gives the characteristic equation

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) + \lambda = -\lambda^3 + 2\lambda = 0.$$

(d) Here we get $\left(-\frac{3}{2} - \lambda\right)^2 - \left(\frac{1}{2}\right)^2 = 0$. Note that if we wanted to solve it, we could multiply this out and then factor, but since we have a difference of squares we could just write $\left(-\frac{3}{2} - \lambda + \frac{1}{2}\right)\left(-\frac{3}{2} - \lambda - \frac{1}{2}\right) = (1-\lambda)(2-\lambda) = 0$ from here. Be careful about "simplifying" an equation—always keep in mind the most useful form of it (for us, factored)! Also, note that this has the same characteristic equation as the matrix in (a), despite being a very different matrix.

(e) Successive cofactor expansions along the middle three rows of

$$\det(A - \lambda I_5) = \begin{vmatrix} -\lambda & 0 & 0 & 0 & 2 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 2 & 0 & 0 & 0 & -\lambda \end{vmatrix}$$

gives the determinant as $(-\lambda)^3 \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = (-\lambda)^3\left((-\lambda)^2 - (2)^2\right) = (-\lambda)^3\left(\lambda^2 - 4\right)$. Thus, our characteristic equation is $-\lambda^3\left(\lambda^2 - 4\right) = 0$.

(f) Here we get $\begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - (2)(-1) = \lambda^2 + 1$.

2 What are the eigenvalues of these matrices? What are the dimensions of each eigenspace?

[Note: Again, try to minimize computation—we're not after the eigenspace itself, just its dimension, so you only need to manipulate the matrix into an Echelon Form matrix, not fully solve for its null space.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

(a) The characteristic equation we computed was $(1-\lambda)(2-\lambda)$, so the eigenvalues are 1 and 2. We can see that each will have a one-dimensional eigenspace (there will be one pivot after subtracting the appropriate multiple of the identity).

(b) The characteristic equation we got above was $\lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0$, so the eigenvalues are $\lambda = \pm 1$. We can see that $A \mp I_2$ will have rank 1 (that is, one pivot), so each eigenspace is again of dimension 1.

(c) The characteristic equation we got was $-\lambda^3 + 2\lambda = -\lambda(\lambda^2 - 2) = 0$, so the eigenvalues are $-\sqrt{2}$, 0, and $\sqrt{2}$. Again, we'll find each has a one-dimensional eigenspace.

(d) This has the same characteristic equation as part (a), so also has 1 and 2 as eigenvalues, and we can check that it also has only a one-dimensional eigenspace for each.

(e) Our characteristic polynomial here is $-\lambda^3\left(\lambda^2 - 4\right) = -\lambda^3(\lambda-2)(\lambda+2) = 0$. We thus have eigenvalues 0 and ± 2. We can check that we'll get 4 pivots for both positive and negative two (the three middle rows will be linearly independent, and the top and bottom will be multiples of each other), so they both have one-dimensional eigenspaces. The matrix for eigenvalue 0 clearly has two pivots (the first and fifth columns are the only nonzero ones, and are clearly linearly independent), so has a three-dimensional eigenspace (there are three free variables).

(f) Our characteristic polynomial is $\lambda^2 + 1 = 0$, which has no real solutions, so there are no real eigenvectors.

Under the Hood: You may have noticed we often got one-dimensional eigenspaces. If we know something is an eigenvalue, we know that its eigenspace is *at least* one dimensional, and the eigenspaces for different eigenvalues are distinct except for the zero vector (otherwise A would act on a vector in both by scaling by the different eigenvalues, which would give two different answers!). Thus, if we have n distinct eigenvalues for a matrix in $\mathbb{R}^n$, we know we have found n distinct subspaces, each of which is at least one-dimensional. This means they *have* to be one-dimensional, otherwise $\mathbb{R}^n$ would have more than n dimensions! As it turns out, the characteristic equation gives us information on the maximum size of each eigenspace, through the *multiplicity* of each root (how many times it appears).