Math 4A
Worksheet 5.3
Lecture 11/27/17

Diagonalization

The Punch Line: Eigenvalues and -vectors can be used to factor a matrix in a way that makes computation easier.

Warm-Up What are the eigenvalues of these matrices? What are their eigenspaces?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

Diagonalizing: If the matrix A has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ corresponding to them, then we write $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ for the matrix whose columns are the eigenvectors and D for the matrix with the eigenvalues down the diagonal and zeroes elsewhere. Then

$$AP = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD.$$

If the eigenvectors are linearly independent, then P is invertible, and $A = PDP^{-1}$.

1 Are these matrices diagonalizable? If so, what are P and D?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

Using the Diagonalization: If we have written $A = PDP^{-1}$ with D a diagonal matrix, then we can easily compute the kth power of A as $A^k = PD^kP^{-1}$ (adjacent P and P^{-1} matrices will cancel, putting all of the D matrices together and just leaving the ones on the end).

2 The Fibonacci numbers are a *very* famous sequence of numbers. The first one is $F_1 = 0$, the second is $F_2 = 1$, and from then on out, each number is the sum of the previous two $F_n = F_{n-1} + F_{n-2}$ (this is sometimes used as a simple model for population growth—although it assumes immortality). Since it's annoying to compute F_n if n is very large (we'd have to do a lot of backtracking to get to known values), it would be nice to have a closed form for F_n. We can derive one with the linear algebra we already know!

(a) Since the equation defining F_n in terms of F_{n-1} and F_{n-2} is linear, we can use a matrix equation to represent the situation. In particular, we want a matrix A such that

$$A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

so that we can keep applying A to get further along in the sequence. What is this A?

(b) Since $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = A^2\begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix} = \cdots$, we can find F_n by computing $A^{n-2}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = A^{n-2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (we only need to advance by $n-2$ steps, because the top entry starts at 2). It's easier to raise matrices to powers after we diagonalize them, so find an invertible P and diagonal D so that $A = PDP^{-1}$ (the numbers are a little gross, so don't be alarmed).

2 cont.

(c) Since $A^k = PD^kP^{-1}$, we can write out F_n as the first component of $PD^{n-2}P^{-1}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = PD^{n-2}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (if we wanted to be clever, we could write this as

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} PD^{n-2}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

as the row vector picks out the first component). Use this to write down a formula for F_n (don't worry about multiplying out powers of any terms involving square roots, just leave them as whatever they are)! Nifty!!!

Under the Hood: The right way to think about the matrices P and P^{-1} is as change-of-coordinates matrices to an *eigenbasis*—then the requirement for diagonalizability is that the eigenvectors of A form a basis for the space they're in. Essentially, what we're doing is choosing a clever basis so that A looks like a diagonal matrix in that basis.