Inner Products, Length, and Orthogonality

The Punch Line: We can compute a real number relating two vectors—or a vector to itself—that gives information on both length and angle.

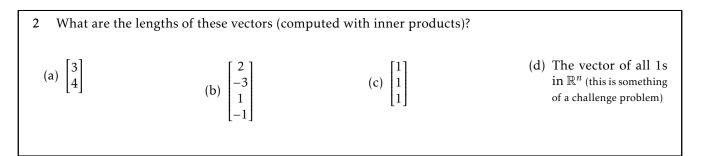
Warm-Up Theorem)?	What are the lengths of these vectors, as found geometrically (using things like the Pythagorean		
(a) $\begin{bmatrix} 3\\ 0 \end{bmatrix}$	(c) $\begin{bmatrix} 1\\1 \end{bmatrix}$	(e) $\begin{bmatrix} -1\\ 2 \end{bmatrix}$	
(b) $\begin{bmatrix} 0\\ -2 \end{bmatrix}$	(d) $\begin{bmatrix} 3\\4 \end{bmatrix}$	(f) $\begin{bmatrix} 1\\1\\3 \end{bmatrix}$	

The Inner Product: If we think about a vector $\vec{v} \in \mathbb{R}^n$ as a $n \times 1$ matrix (a single column), then \vec{v}^T is a $1 \times n$ matrix (a single row, sometimes called a row vector). Then we can multiply \vec{v}^T against a vector (on the left) to get a 1×1 matrix, which we can consider a scalar. This is the idea behind the *inner product* in \mathbb{R}^n , also called the *dot product*: we take two vectors, \vec{u} and \vec{v} , and define their inner product as $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. This corresponds to multiplying together corresponding entries in the vectors, then adding all of the results to get a single number.

1 Find the inner product of the two given vectors:				
(a) $\begin{bmatrix} 1\\1 \end{bmatrix}$ and $\begin{bmatrix} 2\\3 \end{bmatrix}$	(c) $\begin{bmatrix} 1\\-1\\1\\-2 \end{bmatrix}$ and $\begin{bmatrix} 3\\2\\-1\\0 \end{bmatrix}$	(e) $\begin{bmatrix} 0\\0 \end{bmatrix}$ and $\begin{bmatrix} x\\y \end{bmatrix}$		
(b) $\begin{bmatrix} 1\\0 \end{bmatrix}$ and $\begin{bmatrix} 1\\1 \end{bmatrix}$	(d) $\begin{bmatrix} 1\\0 \end{bmatrix}$ and $\begin{bmatrix} 0\\1 \end{bmatrix}$	(f) $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ x \end{bmatrix}$		

Length and Orthogonality: We observe that in \mathbb{R}^2 , the quantity $\sqrt{\vec{v} \cdot \vec{v}}$ is the length of \vec{v} as given by the Pythagorean Theorem. This motivates us to define the length of a vector in *any* \mathbb{R}^n as $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ (encouraged that it also agrees with our idea of length in \mathbb{R}^1 and \mathbb{R}^3). Then the *distance* between \vec{u} and \vec{v} is $\|\vec{u} - \vec{v}\|$, the length of the vector between them.

We also observe that in \mathbb{R}^2 , if \vec{u} and \vec{v} are perpendicular then $\vec{u} \cdot \vec{v} = 0$, and vice versa. To generalize this, we say \vec{u} and \vec{v} are *orthogonal* if $\vec{u} \cdot \vec{v} = 0$ (and indeed, this matches with perpendicularity in three dimensions as well).



3 What is the distance between these two vectors? Are they orthogonal? (a) $\begin{bmatrix} 1\\1 \end{bmatrix}$ and $\begin{bmatrix} 1\\-1 \end{bmatrix}$ (b) $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$ and $\begin{bmatrix} 1\\2\\-3 \end{bmatrix}$ (c) $\begin{bmatrix} 2\\5 \end{bmatrix}$ and $\begin{bmatrix} -2\\-5 \end{bmatrix}$ (d) Two (different) standard basis vectors in \mathbb{R}^n

Under the Hood: This idea of orthogonality can be used to find the collection of *all* vectors which are orthogonal to some given \vec{u} . These are the solutions to the equation $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = 0$. This is just finding the nullspace of the matrix \vec{u}^T , but now it has a nice geometric interpretation. The solution set is a subspace, known as the *orthogonal complement* of \vec{u} .