Orthogonal Sets

The Punch Line: With an inner product, we can find especially nice bases called orthonormal sets.

Warm-Up What are the inner products and lengths of the following pairs of vectors?

(a) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Here we have an inner product $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = (1)(1) + (-2)(0) + (1)(-1) = 0$. The vector lengths are $\sqrt{(1)^2 + (-2)^2 + (1)^2} = \sqrt{6}$ and $\sqrt{(1)^2 + (0)^2 + (-1)^2} = \sqrt{2}$.

(b) Here the inner product is 0 and the length of the all-ones vector is $\sqrt{3}$.

(c) The inner product here is again zero.

(d) The inner product is 0 and both vectors have length $\sqrt{2}$.

(e) The inner product here is 1, the first vector has length 1 and the second has length $\sqrt{2}$.

(f) The inner product here is 2, and the second vector has length $\sqrt{3}$.

Orthogonal and Orthonormal Sets: If the inner product of every pair of vectors in a set $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is zero, we call the set *orthogonal*. In this case, it's a linearly independent set, and so a basis for its span. If there are n vectors in the set, it is a basis for $\mathbb{R}^n$.

If in addition to begin orthogonal, every vector in the set is a *unit vector* (has length 1), we call the set *orthonormal*. Since an orthogonal set is a basis, there is a unique representation of any vector $\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{v}_n$; as it turns out the coefficients $c_i = \frac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i}$. If the set is orthonormal, this means the coefficients are just the inner products with the basis vectors.

1 Are these sets orthogonal? If so, find an orthonormal set by rescaling them.

(a) $\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix}3\\-1\\0\\3\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\4\\-12\end{bmatrix}, \begin{bmatrix}3\\-11\\-7\\-6\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\2\\-2\\\frac{1}{13}\end{bmatrix}, \begin{bmatrix}\sqrt{5}\\1\\0\\0\\-\sqrt{7}\end{bmatrix}, \begin{bmatrix}83\\18\\27\\-1\\0\end{bmatrix}\right\}$

(a) We found that the vectors are orthogonal in the warm-up. They aren't orthonormal, but we can get an orthonormal set by dividing by their lengths. This yields $\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

(b) Again, these vectors are orthogonal but not orthonormal. The orthonormal set is $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

(c) We found previously that these vectors are not orthogonal, because they have nonzero inner products with each other.

(d) Rather than compute the 15 inner products we'd need to check if this set is orthogonal, we can use the knowledge that an orthogonal set is linearly independent. This set is in $\mathbb{R}^5$ and has 6 vectors, so it can't be linearly independent, so it isn't orthogonal. We can also see that the inner product of the second and fifth vectors is $12\sqrt{7} \neq 0$, which also shows it is not orthogonal.

Orthogonal Matrices: In an unfortunate twist of terminology, we call a matrix an *orthogonal matrix* if its columns are an ortho<u>normal</u> set (not just orthogonal like the name might make you think). These matrices are precisely those matrices U where $U^T U = I_n$.

2 Are these matrices orthogonal?

(a) $\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(f) The change-of-coordinates matrices to and from an orthonormal set [Challenge problem]

(a) Yes, we've previously found these to be an orthonormal set.

(b) No, although the columns are orthogonal, they are not orthonormal.

(c) Yes, we can check that $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = I_2$.

(d) No, since $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix} \neq I_3$. This shows that even though $U^T U$ is an identity matrix, it's not necessarily true that UU^T is if U is not square (if it's square, the condition means $U^T = U^{-1}$, so it does commute with U).

(e) No, since although the columns are clearly orthogonal, they are not unit vectors.

(f) Yes, since one will have the orthonormal set as its columns, and the other will be the inverse of that matrix, and because for *square* matrices $U^T U = I_n$ implies $UU^T = I_n$ (by the way inverse matrices work), the inverse must also be orthonormal.

Under the Hood: Orthogonal transformations from $\mathbb{R}^n$ to itself are precisely those which do not change inner products (where $(U\vec{u}) \cdot (U\vec{v}) = \vec{u} \cdot \vec{v}$ for all pairs of vectors). This means they do not change the geometry involved (lengths, relative angles, or distances), so they are particularly interesting transformations.

This is an example of an incredibly common pattern in mathematics: when there is some kind of structure (like a vector space structure, or geometric relationships), mathematicians are interested in finding the collection of functions which preserve that structure (linear transformations and transformations by orthogonal matrices, in those two cases). There are also other classes of linear transformations that preserve things like areas (determinant has absolute value 1), or orientation (determinant is precisely 1), or just angles and not lengths (columns are orthogonal but not necessarily orthonormal), and many more.