Math 4A
Worksheet 6.2
Lecture 12/01/17

Orthogonal Sets

The Punch Line: With an inner product, we can find especially nice bases called orthonormal sets.

Warm-Up What are the inner products and lengths of the following pairs of vectors?

(a) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Orthogonal and Orthonormal Sets: If the inner product of every pair of vectors in a set $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is zero, we call the set *orthogonal*. In this case, it's a linearly independent set, and so a basis for its span. If there are n vectors in the set, it is a basis for $\mathbb{R}^n$.

If in addition to begin orthogonal, every vector in the set is a *unit vector* (has length 1), we call the set *orthonormal*. Since an orthogonal set is a basis, there is a unique representation of any vector $\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{v}_n$; as it turns out the coefficients $c_i = \frac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i}$. If the set is orthonormal, this means the coefficients are just the inner products with the basis vectors.

1 Are these sets orthogonal? If so, find an orthonormal set by rescaling them.

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -12 \end{bmatrix}, \begin{bmatrix} 3 \\ -11 \\ -7 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ \frac{1}{13} \end{bmatrix}, \begin{bmatrix} \sqrt{5} \\ 1 \\ 0 \\ 0 \\ -\sqrt{7} \end{bmatrix}, \begin{bmatrix} 83 \\ 18 \\ 27 \\ -1 \\ 0 \end{bmatrix} \right\}$

Orthogonal Matrices: In an unfortunate twist of terminology, we call a matrix an *orthogonal matrix* if its columns are an orthonormal set (not just orthogonal like the name might make you think). These matrices are precisely those matrices U where $U^T U = I_n$.

2 Are these matrices orthogonal?

(a) $\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 \\ -2 & -1 \\ 1 & -1 \end{bmatrix}$

(d) $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(f) The change-of-coordinates matrices to and from an orthonormal set [Challenge problem]

Under the Hood: Orthogonal transformations from $\mathbb{R}^n$ to itself are precisely those which do not change inner products (where $(U\vec{u})\cdot(U\vec{v}) = \vec{u}\cdot\vec{v}$ for all pairs of vectors). This means they do not change the geometry involved (lengths, relative angles, or distances), so they are particularly interesting transformations.

This is an example of an incredibly common pattern in mathematics: when there is some kind of structure (like a vector space structure, or geometric relationships), mathematicians are interested in finding the collection of functions which preserve that structure (linear transformations and transformations by orthogonal matrices, in those two cases). There are also other classes of linear transformations that preserve things like areas (determinant has absolute value 1), or orientation (determinant is precisely 1), or just angles and not lengths (columns are orthogonal but not necessarily orthonormal), and many more.