The Gram-Schmidt Process

The Punch Line: We can turn any basis into an orthonormal basis using a (relatively) simple procedure.

Warm-Up For what choices of the variables are these bases orthogonal? Can they be made orthonormal by choosing variables correctly?

(a) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}4\\y\end{bmatrix},\begin{bmatrix}x\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix},\begin{bmatrix}x\\y\\0\end{bmatrix},\begin{bmatrix}-x\\0\\z\end{bmatrix}\right\}$

(a) The inner product of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}x\\y\end{bmatrix}$ is x, so if $x = 0$ (and we'll want $y \neq 0$ to prevent the zero vector), then they are orthogonal. If in particular $y = 1$, we have an orthonormal set.

(b) The inner product of $\begin{bmatrix}4\\y\end{bmatrix}$ and $\begin{bmatrix}x\\1\end{bmatrix}$ is $4x + y$, so as long as $y = -4x$, we'll have an orthogonal set. The length of the first vector is $\sqrt{4^2 + y^2} \geq \sqrt{4^2} = 4$, though, so we can't make this set orthonormal.

(c) The inner product of $\begin{bmatrix}x\\y\\0\end{bmatrix}$ and $\begin{bmatrix}-x\\0\\z\end{bmatrix}$ is $-x^2$, so if we were to have an orthogonal set, we would need $x = 0$. But then the inner product of the first two vectors would be $\frac{1}{2}y$, so we'd need $y = 0$ to make the inner product zero. However, this would mean the second vector must be zero, which means we won't be able to find an orthogonal basis by choosing variables. If only there were another way...

The Gram-Schmidt Process: Suppose we know $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is a basis for some subspace W we are interested in. We can make an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for the same subspace by repeatedly stripping away the parts of vectors that are not orthogonal to the previous ones.

In particular, we set $\vec{v}_1 = \vec{w}_1$ (there aren't previous vectors that it could be nonorthogonal to). Then we set $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$ (we take off any part of $\vec{w}_2$ that's in the direction $\vec{v}_1$ with a projection). Similarly, we set $\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$ (we have to remove parts in the first *two* directions now). In general, we set

$$\vec{v}_k = \vec{w}_k - \frac{\vec{w}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{w}_k \cdot v_{k-1}}{\vec{v}_{k-1} \cdot v_{k-1}}\vec{v}_{k-1}$$

(subtracting off the projection onto all previous vectors in the basis we are constructing).

1 Apply the Gram-Schmidt Process to the following (ordered) bases:

(a) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-3\end{bmatrix}, \begin{bmatrix}17\\-3\\1\end{bmatrix}\right\}$

(a) Here, we choose $\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$. Then the projection of the second vector onto $\vec{v}_1$ is $\begin{bmatrix}1\\0\\0\end{bmatrix}$, so we choose $\vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$. The projection of the third onto $\vec{v}_1$ is also $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and onto $\vec{v}_2$ is $\begin{bmatrix}0\\1\\0\end{bmatrix}$, so we choose $\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Thus, when we apply the Gram-Schmidt Process to this basis, we get the standard basis out.

(b) Here, we choose $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Then $\vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} - \begin{bmatrix}1/6\\1/6\\-1/3\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$. This set is dramatically different than the standard basis—the order of the basis you start with has a large impact on the Gram-Schmidt Process. To put them all together, our orthogonal basis is then $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\}$.

(c) Here we get $\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Then we get $\vec{v}_2 = \begin{bmatrix}2\\0\\-3\end{bmatrix} - \frac{10}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5/2\\1\\-3/2\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}17\\-3\\1\end{bmatrix} - \frac{14}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{76/2}{19/2}\begin{bmatrix}5/2\\1\\-3/2\end{bmatrix} = \begin{bmatrix}6\\-9\\4\end{bmatrix}$. To put them all together, our orthogonal basis is $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5/2\\1\\-3/2\end{bmatrix}, \begin{bmatrix}6\\-9\\4\end{bmatrix}\right\}$.

Orthonormal Bases: After applying the Gram-Schmidt Process, it's easy to get an orthonormal basis—just rescale the results. It's important to note that the rescaling can be done right after subtracting off the projections onto the previous vectors, but shouldn't be done before doing so, as subtracting vectors changes lengths (it won't harm the process, but you won't get unit vectors out of it).

2 Find the orthonormal bases from the results of Problem 1.

(a) The standard basis is already orthonormal, so here we're done.

(b) Here, we find the norm of the first vector is $\sqrt{3}$, of the second $\sqrt{\frac{2}{3}}$, and of the third $\sqrt{\frac{1}{2}}$. Thus, our orthonormal basis is $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \sqrt{\frac{3}{2}}\begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}, \sqrt{2}\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\}$.

(c) Here, the resulting orthonormal basis is $\left\{\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}, \frac{1}{\sqrt{38}}\begin{bmatrix}5\\2\\-3\end{bmatrix}, \frac{1}{\sqrt{133}}\begin{bmatrix}6\\-9\\4\end{bmatrix}\right\}$.