Math 4A
Key 1.1

Systems of Linear Equations

The Punch Line: We can solve systems of linear equations by manipulating a matrix that represents the system.

Warm-Up: Which of these equations are linear?

(a) $y = mx + b$, with x and y as variables

(b) $(y - y_0) + 4(x - x_0) = 0$, with x and y as variables

(c) $4x + 2y - 9z = 12$, with x, y, and z as variables

(d) $x_1^2 + x_2^2 = 1$, with x_1 and x_2 as variables

(e) $a^2x + 3b^3y = 6$, with x and y as variables

(f) $a^2x + 3b^3y = 6$, with a and b as variables

(a) This system is linear, because it can be written $-mx + y = b$.

(b) This system is linear, because it can be written $4x + y = (y_0 + 4x_0)$.

(c) This system is linear, because it is already in the standard form.

(d) This system is not linear, because it involves squared variables, so it can't be put into the standard form.

(e) This system is linear with these variables, because it is already in standard form (it is okay for coefficients to be squared).

(f) It is not linear with these variables, because it can't be written in the standard form (the variables have operations done to them that aren't simply scaling or shifting).

The Setup: When we have a linear system of equations, we can make an *augmented matrix* representing the system by arranging the coefficients on the left side of the matrix (keeping them in the same order for each equation, and writing a 0 whenever a variable is missing from one of the equations), and the constants on the right side. This is often easiest to do when each equation is written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, so it can help to rewrite equations like this if they are given to you differently.

1. Write down an augmented matrix representing these linear systems.

(a) The system for x_1 and x_2 given by

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - 2x_2 &= 1 \end{aligned}$$

(b) The system for x, y, and z given by

$$\begin{aligned} x - 2y + z &= 0 \\ x + y &= 2 \\ y - z &= 1 \end{aligned}$$

(c) The system for x and y given by

$$\begin{aligned} y &= 4 - x \\ x + 1 &= 2y + 2 \end{aligned}$$

(a) $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -2 & 1 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right]$

(c) $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -2 & 1 \end{array}\right]$

2. Write down a linear system of equations represented by these augmented matrices.

(a) $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(a) The system in x_1 and x_2 given by $$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_2 - 3x_1 &= 0 \end{aligned}$$

(b) The system in x_1, x_2, and x_3 given by $$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}$$

(c) The system in x_1, x_2, and x_3 given by $$\begin{aligned} x_1 + 2x_2 - 4x_3 &= -1 \\ 0 &= 0 \end{aligned}$$ (the second equation may be omitted)

The Execution: Once we have a matrix representing a linear system of equations, we can use *elementary row operations* on the matrix to find equivalent system of equations. These operations are

1) Replacement: Replace a row with itself plus a multiple of a different row,

2) Interchange: Switch the order of two rows,

3) Scaling: Multiply everything in the row by the same constant (other than 0).

The goal is to use these three operations to find an equivalent system of equations that is easier to solve.

3. Solve each of these linear systems of equations.

(a) The two variable system given by

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - 2x_2 &= 1 \end{aligned}$$

(b) The three variable system given by

$$\begin{aligned} x - 2y + z &= 0 \\ x + y &= 2 \\ y - z &= 1 \end{aligned}$$

(c) The three variable system given by

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}$$

(a) We know from part **1.** that the augmented matrix of this system is $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -2 & 1 \end{array}\right]$. If we replace Row 2 with itself minus Row 1 (that is, itself plus -1 times Row 1), we get the matrix $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -3 & -3 \end{array}\right]$. Then, we can scale Row 2 by $\frac{-1}{3}$ to get the matrix $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 1 \end{array}\right]$. Finally, we can replace Row 1 with itself minus Row 2 to get $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$. This represents the linear system of equations $\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \end{aligned}$, which we can clearly see has the (unique) solution $s_1 = 3$, $s_2 = 1$.

(b) The augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right]$. If we replace Row 2 with itself minus Row 1, we get $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right]$. We can then interchange Row 2 and Row 3 to get $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -1 & 2 \end{array}\right]$. Then we can replace Row 3 with itself minus three times Row 2, to get $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right]$. We scale Row 3 by $\frac{1}{2}$ to get $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} \end{array}\right]$. The next step is to replace Row 2 with itself plus Row 3, which gives $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} \end{array}\right]$. We replace Row 1 with itself plus twice Row 2 to get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} \end{array}\right]$. Finally, we replace Row 1 with itself minus Row 3 to get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} \end{array}\right]$. From this, we can see that $\left(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}\right)$ is the solution to this system.

(c) The augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$. We replace Row 2 with itself minus Row 1 to get $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -3 \\ 1 & 0 & -1 & 0 \end{array}\right]$, then scale Row 2 by $\frac{-1}{3}$ to get $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right]$. We replace Row 1 with itself minus

Row 2 to get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right]$. We replace Row 3 with itself minus Row 1 to get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right]$. Then, we scale Row 3 by $\frac{-1}{2}$ to get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$. Finally, we replace Row 1 with itself minus Row 3 to get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$. From this, we see that $s_1 = s_2 = s_3 = 1$ is the solution to this system.

Under the Hood: Why do elementary row operations result in equivalent systems of equations? Each equation is just some true statement about the solutions, and the system of equations is a collection of true statements that together give us enough information to figure out exactly what the solutions are. Elementary row operations are tools we use to make new true statements that contain the same amount of information about the solutions as the old ones. We know the new statements contain the same amount of information because they're reversible—if we started with them, we could do a different series of operations to get the original system. Come see me if you want to talk about why we are sure they make true statements—or try to prove it on your own!