Vector Equations

The Punch Line: Vector equations allow us to think about systems of linear equations as geometric objects, and are an efficient notation to work with.

Warm-Up: Sketch the following vectors in $\mathbb{R}^2$:

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

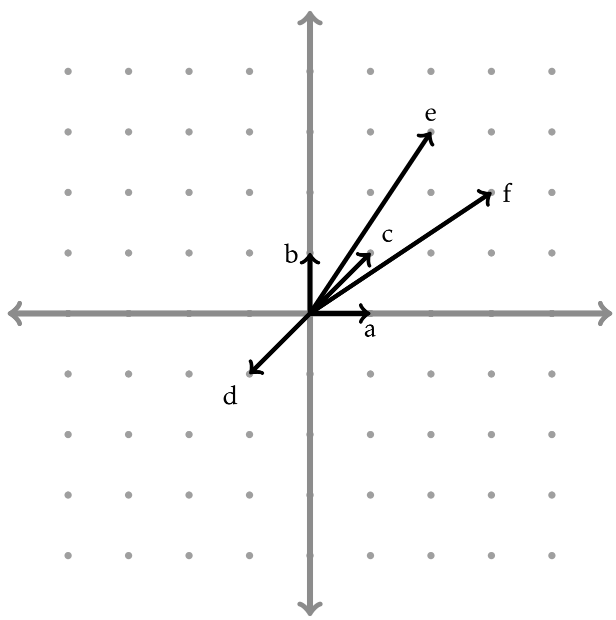
(b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(f) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$



Linear Combinations: A *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with *weights* $w_1, w_2, \ldots, w_n$ is the vector $\mathbf{y}$ defined by

$$\mathbf{y} = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n.$$

That is, it's a sum of multiples of the vectors. Geometrically, it corresponds to stretching each vector $\mathbf{v}_i$ (where i is one of $1, 2, \ldots, n$) by the weight w_i, then laying them end to end and drawing $\mathbf{y}$ to the endpoint of the last vector.

1 Compute the following linear combinations:

(a) $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$

(b) $(-1)\begin{bmatrix}1\\1\end{bmatrix}$

(c) $\begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix}$

(d) $\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix}$

(e) $\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$

(f) $4\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix} - 2\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix} + 3\begin{bmatrix}\frac{2}{9}\\2\end{bmatrix}$

Think about what each of these linear combinations mean geometrically (try sketching them).

(a) Addition of vectors is componentwise, so this linear combination yields $\begin{bmatrix}1\\1\end{bmatrix}$.

(b) Multiplication of a number and a vector (called *scalar multiplication* because the number is acting to scale the vector) is also componentwise, so this is $\begin{bmatrix}-1\\1\end{bmatrix}$.

(c) Applying the rules in sequence, we get $\begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}-3\\-2\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$.

(d) The answer here is $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

(e) This one is $\begin{bmatrix}1\\1\end{bmatrix}$.

(f) Finally, $\begin{bmatrix}4\\6\end{bmatrix}$.

Span: The *span* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the set of all linear combinations of them. If $\mathbf{x}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then we will be able to find some weights $w_1, w_2, \ldots, w_n$ to make the linear combination using those weights result in $\mathbf{x}$:

$$w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n = \mathbf{x}.$$

Often, we are interested in determining if a given vector is in the span of some set of other vectors. In particular, a system of linear equations has a solution precisely when the rightmost column of the augmented matrix is in the span of the columns to the left of it. This means a system of linear equations is equivalent to a single vector equation.

2 Determine if $\mathbf{x}$ is in the span of the given vectors:

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

(b) $\mathbf{x} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\mathbf{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

If it is, describe the linear combination that yields it.

(a) To check this, we write down the vector equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{x},$$

which says "the linear combination with weights a_1 and a_2 of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\mathbf{x}$". If $\mathbf{x}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, then this equation will have a solution. We can write it out in components to see that this is equivalent to the system of linear equations

$$\begin{aligned} a_1 - 2a_2 &= 1 \\ a_1 &= 1 \\ a_1 + 2a_2 &= 1. \end{aligned}$$

By computing the Reduced Echelon Form of the augmented matrix of this system, we can identify any solutions, if they exist. However, the REF is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since the last column has a pivot entry, we can see that this system is inconsistent. This means that the system of linear equations, and therefore the vector equation, does not have a solution. This means no linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ yields $\mathbf{x}$, so it is not in their span.

(b) By following the above procedure, we can find that $\mathbf{x}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, with weights $a_1 = 13$ and $a_2 = -1$.

(c) Similarly, we find here that $\mathbf{x}$ is in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Our REF is $\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$, so we see that we have a free variable x_3, so there are infinitely many linear combinations that give $\mathbf{x}$. In particular, if $a_1 = 5 + a_3$ and $a_2 = -4 - 2a_3$ (and a_3 is anything) we have $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{x}$.

Under the Hood: The span of a collection of vectors is essentially the set of all vectors that can be constructed using the members of the collection as components. This means that if a vector is *not* in the span of the collection, it has some additional component that's different from everything in the collection.