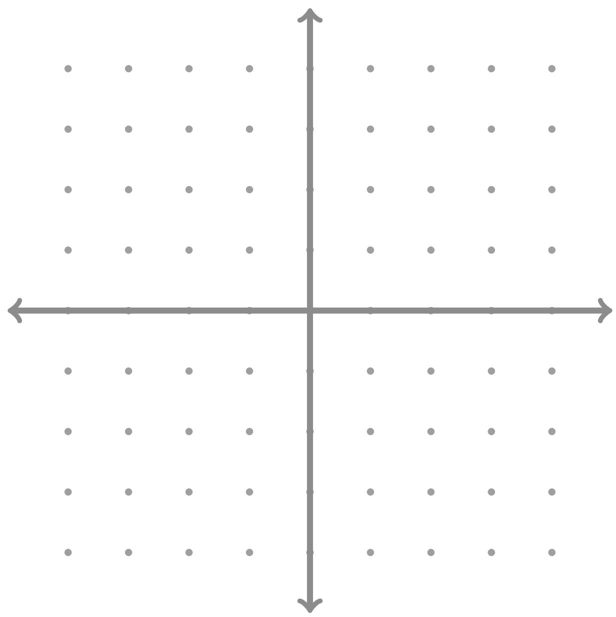
Solution Sets of Linear Systems

The Punch Line: There is a geometric interpretation to the solution sets of systems 0f linear equations, which allows us to explicitly describe them with *parametric equations*.

Warm-Up: Draw the line in $\mathbb{R}^2$ defined by $y = 3 - 2x$.



Verify that $x(t) = 1 + t$ and $y(t) = 1 - 2t$ satisfy the equation $y(t) = 3 - 2x(t)$ for all t, and plot $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ for $t = -1$, 0, and 1.

Homogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{0}$ is called *homogeneous*. It always has the solution $\vec{x} = \vec{0}$, which is called the *trivial solution*. Any other solution is called a *nontrivial solution*; nontrivial solutions arise precisely when there is at least one free variable in the equation.

If there are m free variables in the homogeneous equation, the solution set can be expressed as the span of m vectors:

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_m\vec{v}_m.$$

This is called a *parametric equation* or a *parametric vector form* of the solution. A common parametric vector form uses the free variables as the parameters s_1 through s_m.

1 Find a parametric vector form for the solution set of the equation $A\vec{x} = \vec{0}$ for the following matrices A:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 4 & -4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Nonhomogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$ is called *nonhomogeneous*. As we've seen, a nonhomogeneous system may be inconsistent and fail to have solutions. If it does have a solution, though, we can find a parametric form for them as well as in the homogeneous case. Here, we express the solutions as $\vec{x} = \vec{p} + \vec{v}_h$, where $\vec{p}$ is some particular solution to the nonhomogeneous system (which we can get by picking simple values for the parameters, such as taking all free variables to be zero), and $\vec{v}_h$ is a parametric form for the solution to the *homogeneous* equation $A\vec{v}_h = \vec{0}$.

2 If possible, find a parametric vector form for the solution set of the nonhomogeneous equation $A\vec{x} = \vec{b}$ for the following matrices A and vectors $\vec{b}$ (otherwise explain why it is impossible):

(a) $\begin{bmatrix} 1 & 2 \end{bmatrix}; \begin{bmatrix} 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Under the Hood: Why do the solution sets to nonhomogeneous solutions have a "homogeneous part"? Imagine we are given two vectors, $\vec{x}_1$ and $\vec{x}_2$, and we're assured that $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$. That is, we have two solutions to the nonhomogeneous equation. We can take the difference between these two equations to see that $A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b}$. A property of matrix-vector multiplication lets us write the left-hand side as $A(\vec{x}_1 - \vec{x}_2)$, while the right-hand side is clearly $\vec{0}$, so we're left with the equation $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. That is, we've just shown the *difference* between two solutions to the nonhomogeneous equation is always a solution to the homogeneous equation with the same matrix!