Applications of Linear Systems

The Punch Line: Linear systems of equations can describe many interesting situations.

Set-Up: In a situation you can model with linear equations, there will be a number of *constraints*: things which must be equal because of the laws governing what's going on (e.g., laws of physics, economic principles, or definitions of quantities and the values you observe for them). These will give you the equations that you can solve to get information about the variables you care about

1 In the past three men's soccer games, the Gauchos averaged $\frac{5}{3}$ goals per game. They scored the same number of goals in the most recent two games, but three games ago they scored an additional two goals. How many points did they score in each game?

(This problem was actually true as of the 11^{th}, but even if you know the scores, it's probably helpful to set up the system and see how they come out of the equations.)

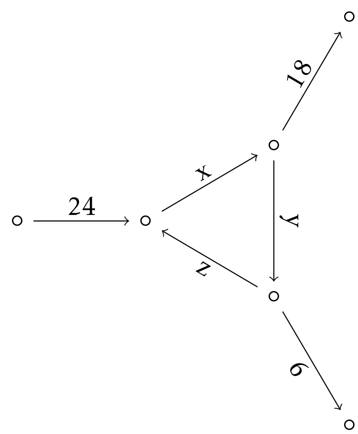
We'll call the number of goals in the past three games g_1, g_2, and g_3, with g_1 being the most recent. That the average is $\frac{5}{3}$ means that $\frac{1}{3}(g_1 + g_2 + g_3) = \frac{5}{3}$, which we can simplify to $g_1 + g_2 + g_3 = 5$. That the past two games had the same number of goals means $g_1 = g_2$, which we can put into the more standard form (with variables on one side and constants on the other) of $g_1 - g_2 = 0$. That three games ago there were two more goals can be represented with the equation $g_3 = g_2 + 2$ (we could equally well have used g_1, or even $\frac{1}{2}(g_1 + g_2)$—any quantity which is equal to their shared value). This can be written as $g_3 - g_2 = 2$. Thus, we have the system of linear equations

$$\begin{cases} g_1 + g_2 + g_3 & = 5 \\ g_1 - g_2 & = 0 \\ g_3 - g_2 & = 2. \end{cases}$$

The REF of the augmented matrix of the system is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, so the unique solution is

$$\vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

2 Suppose you're watching a bike loop on campus and writing down the net number of bicycles travelling through each part of the loop (the number of bikes going one direction minus the number going the other direction). You're able to observe how many net bikes per minute enter and leave through each of the three spokes, but aren't able to count well inside the loop. Luckily, you can use linear algebra to learn about how many net bikes per minute travel through each part of the loop (which is to say, find all solutions for x, y, and z that are consistent with the rest of the information about the problem)!



Here we're going to appeal to a principle I'd call "conservation of bikers": assuming that the (net) number of bicycles entering a junction of paths equals the (net) number of bicycles exiting that junction (this is an example of a "conservation law", which is a broad class of physical laws that often generate constraints that can be used in modelling systems). Applying this rule to each of the three junctions in the above diagram gives the system of equations

$$\begin{cases} 24+z & = x \\ x & = 18+y \\ y & = 6+z. \end{cases}$$

Rearranging these into standard form (with variables on one side and constants on the other) gives

$$\begin{cases} x-z & = 24 \\ x-y & = 18 \\ y-z & = 6. \end{cases}$$

Finding the REF of the augmented matrix of this system yields $\begin{bmatrix} 1 & 0 & -1 & 24 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We extract the parametric vector form of the solution from this, obtaining

$$\vec{x} = \begin{bmatrix} 24 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This means the set of solutions has an infinite number of possibilities, one for each possible (net) number of bikes travelling along path z.

3 (Example 1 in Section 1.6) In an *exchange model* of economics, an economy is divided into different sectors which depend on each others' products to produce their output. Suppose we know for each sector its total output for one year and exactly how this output is divided or "exchanged" among the other sectors of the economy. The total dollar (or other monetary unit) value of each sector's output is called the *price* of that output. There is an *equilibrium price* for this kind of model, where each sectors income exactly balances its expenses. We wish to find this equilibrium.

Suppose we have an economy described by the following table:

Distribution of output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
0.0	0.4	0.6	Coal
0.6	0.1	0.2	Electric
0.4	0.5	0.2	Steel

If we denote the price of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S respectively, what is the equilibrium price (or describe them if there is more than one).

We're trying to balance the input into each sector with the output, which is represented by the price variable for that sector. The input is the proportion of each sector's output devoted to the purchasing sector (as given in the table). This means we get the system of equations

$$\begin{cases} p_C &= 0.4p_E + 0.6p_S \\ p_E &= 0.6p_C + 0.1p_E + 0.2p_S \\ p_S &= 0.4p_C + 0.5p_E + 0.2p_S. \end{cases}$$

We write them in standard form (moving all variables to one side), and for computational convenience multiply each resulting equation by 10 to deal with integers rather than decimals, noting that this won't change our answer. This gives the homogeneous system of equations

$$\begin{cases} 10p_C - 4p_E - 6p_S &= 0 \\ -6p_C + 9p_E - 2p_S &= 0 \\ -4p_C - 5p_E + 8p_S &= 0. \end{cases}$$

The REF of the augmented matrix of this system is $\begin{bmatrix} 1 & 0 & -\frac{31}{33} & 0 \\ 0 & 1 & -\frac{28}{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This means that our solutions set has the parametric vector form $\vec{p} = t\begin{bmatrix} 31 \\ 28 \\ 33 \end{bmatrix}$ (where the parameter $t = \frac{1}{33}p_S$ was chosen for integer entries in the vector). This means that the equilibrium is reached whenever the ratios between the prices are precisely 31 : 28 : 33 (regardless of the actual magnitude; if all prices doubled, say, there would still be an equilibrium).

Under the Hood: When can we use linear equations to model something? The basic setup of a linear system involves a collection of quantities that we know are equal to known values (or each other), and a collection of variables. We can use a linear system when the way the quantities depend on changes to the variables is independent of the actual values of the variables (adding the same amount to a variable changes each quantity in the same way, no matter what value any of the variables have).