Math 4A
Key 1.7

Linear Independence

The Punch Line: *Linear independence* is a property describing a collection of vectors whose span is "as big as it can be"—no vector is redundant, but each one contributes a "different direction" to the span.

Warm-Up: Are each of these situations possible?

(a) The vectors $\{\vec{u}, \vec{v}\}$ in $\mathbb{R}^2$ span all of $\mathbb{R}^2$.

(b) The vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span all of $\mathbb{R}^3$.

(c) The vectors $\{\vec{x}, -\vec{x}\}$ span all of $\mathbb{R}^2$.

(d) The vectors $\{\vec{u}, \vec{v}\}$ span all of $\mathbb{R}^3$.

(e) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ intersect only at $\vec{0}$.

(f) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ are both planes and intersect only at $\vec{0}$.

(a) Yes, two vectors can span a plane.

(b) Yes, four vectors can span 3-space.

(c) No, they are both on the same line, so their span is that line.

(d) No, two vectors can span at most a plane.

(e) Yes, but only if at least one of the spans is just a line and not a whole plane.

(f) No, any two planes in $\mathbb{R}^3$ that include $\vec{0}$ (or any specific point in common) must intersect on at least a line.

Tests for Linear Independence: That the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent means that if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, the only possibility is that $c_1 = c_2 = \cdots = c_n = 0$. That is, the only solution to the homogeneous matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

is the trivial solution: the zero vector. If there's only one vector in the set, it is linearly independent unless it happens to be $\vec{0}$. If there's only two vectors, they are linearly independent unless one is a multiple of the other (including $\vec{0}$, which is 0 times any vector). If a subset of the vectors is linearly dependent, the whole set is. Finally, a set of vectors is linearly dependent if (and only if) at least one vector is in the span of the others.

1 Are these sets linearly dependent or independent? Why?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix} \right\}$

(a) No, the second vector is -1 times the first.

(b) No, the third vector is $\frac{1}{4}$ times the first, plus there are three vectors in $\mathbb{R}^2$.

(c) Yes, and we can check by observing there are no free variables in the REF of the coefficient matrix of the homogeneous equation described above.

(d) No, the third vector is the sum of the first and -2 times the second.

2 Are each of these situations possible?

(a) You have a set of vectors that spans $\mathbb{R}^3$. You remove two of them, and the set of vectors left behind is linearly independent.

(b) You have two sets of vectors in $\mathbb{R}^6$. One has four vectors, and one has two vectors, and both sets are linearly independent. When you put both sets together, the resulting set of six vectors is linearly dependent.

(c) You have a set of three vectors which span $\mathbb{R}^3$, but it is linearly dependent.

(d) You have a linearly dependent set of three vectors in $\mathbb{R}^2$. If you remove any one of them, the other pair do not span $\mathbb{R}^2$.

(a) Yes, for example, if we started with $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and removed the last two.

(b) Yes, for example, if each vector in the set of two was a multiple of one from the set of four.

(c) No, because if the set is linearly dependent, one vector would be in the span of the other two, and the span of two vectors is a plane, while $\mathbb{R}^3$ is a 3-space.

(d) Yes, for example if the three vectors are all multiples of each other, so their span is a line.