Characterizing Invertible Matrices

The Punch Line: There are many equivalent conditions to determine if a matrix is invertible, and describe properties of ones that we know are invertible.

Warm-Up: How big is the solution set of the homogeneous equation with these matrices (is it finite or infinite? what is its dimension?)? How about the span of their columns?

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

(a) The solution set to the homogeneous is finite, and consists of just the origin. The columns span $\mathbb{R}^3$.

(b) The solution set has one free variable, so is infinite and one-dimensional. There are three pivot rows, so the columns span a three-dimensional space.

(c) The solution set has two free variables, so is infinite and two-dimensional. There are two pivots, so the span of the columns is two-dimensional as well.

Matrix Conditions: Our first definition for invertible matrices states that A is invertible if some other matrix C makes the equations $AC = I_n$ and $CA = I_n$ simultaneously true. We can show that if A is invertible, so is A^T, that the inverse of A^T is C^T, and that if either one of the two equations in the definition is true, the other one must be as well, by playing around with transposing the equations.

1 Can each of these things happen? Do they have to be true?

(a) A is invertible and the matrix C with $AC = I_n$ is also invertible

(b) C is an inverse to both A and A^T (that is, $CA = I_n$ and $CA^T = I_n$)

(c) $CA = I_n$ and $AD = I_n$, but $C \neq D$

(d) $ABCD = I_n$, but $AB \neq I_n$ and $CD \neq I_n$

(a) This is always true—if A is invertible, its inverse (C in the above equation, often written A^{-1}) is also invertible, with inverse A. That is, *invertible matrices come in pairs*.

(b) This is possible, but requires that $C = C^T$, and implies $A = A^T$ (because we can just transpose the equations, and inverses are unique).

(c) This is not possible: if we take the transpose of the right equation, we get $D^T A^T = I_n^T = I_n$, but we know that C^T is the inverse for A^T, and inverses are unique.

(d) This is possible—it is saying that AB and CD are a pair of inverse matrices. Consider the case where all of them are $\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The two products become $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which aren't identity matrices, but multiply to one.

Equation Conditions: We also have conditions based on the homogeneous and inhomogeneous equations involving the matrix. We know A is invertible if it is square ($n \times n$) and its columns span $\mathbb{R}^n$ or are linearly independent (for square matrices these are equivalent, though not in general). That is, it has n pivots (so its EF's have pivots in every column, and its REF is I_n) or no free variables. That is, the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ (which will turn out to be unique) or $A\vec{x} = \vec{0}$ has *only* the trivial solution.

2 Are these matrices invertible?

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & -4 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Yes—the columns are linearly independent, because the only way to get the right third component is to use the third column, then the only way to get the right second component is to set the second column to compensate, and so on.

(b) Yes—reducing to Echelon Form such as $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ shows there are 4 pivots.

(c) No—the third column is the sum of the second plus twice the first, so they are not linearly independent.

(d) No—there is no fourth component to any of the vectors, so $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

Why do the columns have to be linearly independent and span all of $\mathbb{R}^n$? If they were not linearly independent, there would be multiple solutions to $A\vec{x} = \vec{b}$, so we couldn't define $A^{-1}\vec{b} = \vec{x}$—we wouldn't know which to choose! And if they did not span $\mathbb{R}^n$, then there would be some $\vec{b}$ outside their span where we couldn't find any $\vec{x}$ so that $A\vec{x} = \vec{b}$—we'd again have a problem defining the inverse, but this time instead of having to many possible answers $\vec{x}$, we wouldn't have any!