Subspaces of $\mathbb{R}^n$

The Punch Line: Some parts of $\mathbb{R}^n$ behave exactly like copies of $\mathbb{R}^m$ (where m is smaller than n) that are sitting inside of the larger space.

Warm-Up

(a) In $\mathbb{R}^3$, if you add two vectors in the $y = 0$ plane, is the result guaranteed to be in the $y = 0$ plane?

(b) Is the answer the same or different for the $y = 1$ plane?

(c) In $\mathbb{R}^2$ if you take two vectors with x component greater than 1 and add them, is the result guaranteed to have an x component greater than 1?

(d) In $\mathbb{R}^2$, if you have a vector with x component greater than 1 and take a scalar multiple of it, is the result guaranteed to have an x component greater than 1?

(e) In $\mathbb{R}^2$, if you have two vectors that each lie on one of the axes, is their sum guaranteed to lie on an axis?

(f) In $\mathbb{R}^2$, if a vector lies on one of the axes and you take a scalar multiple of it, is the result guaranteed to be on one of the axes?

(a) Yes—since the y component of each vector is zero, the y component of their sum is $0 + 0 = 0$, so the sum is in the $y = 0$ plane.

(b) No—in fact, it never does, as the y component of each vector is 1, so the y component of their sum is $1+1 = 2$, so the sum is on the $y = 2$ plane rather than the $y = 1$ plane.

(c) Yes—the x component of the sum will be the sum of the x components, and the sum of two numbers greater than one is greater than one.

(d) No—the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ has x component greater than 1, but $\frac{1}{4}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ does not.

(e) No—the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ lie on the x and y axes, respectively, but their sum is on neither axis.

(f) Yes—scaling just changes the length, not the direction, of a vector, so one that started on an axis will stay on that same axis after scaling.

The Definition: A *subspace* of $\mathbb{R}^n$ is a subset[1] H that satisfies the following three properties:

i) H contains the vector $\vec{0}$

ii) If the vectors $\vec{u}$ and $\vec{v}$ are both in H, then so is $\vec{u}+\vec{v}$

iii) If the vector $\vec{u}$ is in H, then for any real number c the vector $c\vec{u}$ is in H

If we want to test if a subset H is a subspace, we just have to see if these properties hold for it.

1 Are these things subspaces?

(a) The subset $\{\vec{0}\}$ in any $\mathbb{R}^n$

(b) The $y=0$ plane in $\mathbb{R}^3$

(c) The $y=1$ plane in $\mathbb{R}^3$

(d) The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 1$

(e) The axes in $\mathbb{R}^2$

(f) The set of solutions to the matrix equation $A\vec{x}=\vec{0}$

(g) The set of solutions to the matrix equation $A\vec{x}=\vec{b}$ (where $\vec{b}\neq\vec{0}$)

(h) The span of the columns of the matrix A (for any matrix; for concreteness, feel free to think about 3×3 matrices in particular, although it is true for $m\times n$ matrices for any m and n)

(a) Yes—adding only the zero vector and scaling the zero vector don't do anything to it, and obviously $\vec{0}$ is in $\{\vec{0}\}$—it's the *only* thing in it!

(b) Yes—$\vec{0}$ is in the $y=0$ plane, we saw that property ii) held in the warm-up, and scaling a vector with y component zero won't make the y component nonzero, so property iii) holds as well. Since all the properties are true, the $y=0$ plane is a subspace of $\mathbb{R}^2$.

(c) No—in the warm-up we saw that property ii) doesn't work, but also iii) fails (scaling by anything but 1 changes the y component), and in fact $\vec{0}$ isn't in the $y=1$ plane so i) fails as well! Of course, as soon as we notice that *any* of these properties failed, we knew that the $y=1$ plane is not a subspace.

(d) No—in the warm-up we saw that property iii) fails, and of course so does i). In this case, property ii) does *not* fail, even though it is not a subspace.

(e) No—in the warm-up we saw property ii) fails, so this is not a subspace. In this case, property i) and iii) are both true, so it's important to check all three properties.

(f) Yes—this is actually a very important subspace, called the *null space* of A. We know that $\vec{0}$ is a solution to the equation $A\vec{x}=\vec{0}$ because the product of any matrix with the zero vector is the zero vector. Since multiplication by a matrix is a linear transformation, if we know $A\vec{u}=\vec{0}$ and $A\vec{v}=\vec{0}$, then we also know $A(\vec{u}+\vec{v})=A\vec{u}+A\vec{v}=\vec{0}+\vec{0}=\vec{0}$, and similarly $A(c\vec{u})=cA\vec{u}=\vec{0}$, so properties ii) and iii) hold as well.

(g) No—the quickest way to see this is to consider that $A\vec{0}=\vec{0}\neq\vec{b}$, but in fact properties ii) and iii) fail as well.

(h) Yes—this is another very important subspace, known as the *column space* of A. The vector $\vec{0}$ is a linear combination of the columns of any matrix A (just use all weights zero), so i) holds. If $\vec{u}$ and $\vec{v}$ are linear combinations of the columns of a matrix A, then so is $\vec{u}+\vec{v}$ (use the sum of the weight from $\vec{u}$ and the one from $\vec{v}$ on each column), and so is $c\vec{u}$ (use c times the weights from $\vec{u}$). This shows that this is a subspace.

[1] A *subset* of $\mathbb{R}^n$ is just some collection of vectors in $\mathbb{R}^n$.

A Basis: A *basis* for a subspace is a linearly independent set whose span is precisely that subspace. To check if a collection of vectors is a basis for a subspace H, we can put the vectors as the columns of a matrix B. Then the requirement that it is linearly independent is satisfied precisely if every *column* is a pivot column (equivalently, there are no free variables), and the requirement that the span is H is satisfied if the equation $B\vec{x} = \vec{b}$ has a solution precisely when $\vec{b} \in H$. In the special case that H is all of $\mathbb{R}^n$, these conditions are equivalent to B being invertible.

2 Are the following sets of vectors bases for the specified subspaces? (You may assume that it is indeed a subspace.)

(a) The set $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ for the subspace $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

(b) The set $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}$ for the "subspace" $\mathbb{R}^2$

(c) The set $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ for the subspace Span $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

(d) The set $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\2\\-1\end{bmatrix}\right\}$ for the subspace of $\mathbb{R}^3$ consisting of all vectors whose components sum to zero.

(e) The set $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ for the subspace Span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

(a) No—any set containing the zero vector is linearly dependent, but a basis must be linearly independent.

(b) Yes—they are linearly independent (two vectors are linearly dependent if and only if one is a multiple of the other), and since $B = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ is invertible, the equation $B\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^2$.

(c) No—they are linearly independent, but their span is all of $\mathbb{R}^2$, while the subspace is not all of $\mathbb{R}^2$.

(d) Yes—the two vectors are linearly independent. The equation $B\vec{x} = \vec{b}$ has the augmented matrix

$$\left[\begin{array}{cc|c}1 & -1 & b_1\\0 & 2 & b_2\\-1 & -1 & b_3\end{array}\right].$$

When row reducing this, we see that we will get a contradiction unless $b_1 + b_2 + b_3 = 0$ (and that if that equation is true the system is consistent). That is, $B\vec{x} = \vec{b}$ has a solution precisely when the components of $\vec{b}$ sum to zero, as desired.

(e) The matrix

$$B = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 3 & -1\end{bmatrix}$$

has REF

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}.$$

This has a free variable, so the set of vectors is not linearly independent, so can't be a basis, even though its span is clearly the subspace in question.

What's special about a subspace? It "looks like" $\mathbb{R}^m$ living inside $\mathbb{R}^n$. Eventually, we want to capitalize on this to break complicated descriptions into simpler ones. For example, we might be excited to discover that for a part of $\mathbb{R}^{37}$ that looks like $\mathbb{R}^2$, a particularly nasty linear transformation works just like rotation (even if it's hard to describe elsewhere). Subspaces are precisely the parts of $\mathbb{R}^n$ that work nicely with things like linear equations and transformations.