Determinants!

The Punch Line: We can compute a value from the entries of a matrix to get yet *another* way of characterizing invertible matrices. ***SPOILER ALERT***: The determinant will also give us a variety of other useful pieces of information in understanding a matrix and its associated linear transformation!

Warm-Up: Are these matrices invertible? Are there conditions that make them so or not so depending on certain values? Try to answer without reducing them to REF (and in general, with as few computations as possible).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

(a) This is not invertible—the columns are linearly dependent, violating one of the 10 conditions from last time.

(b) This is invertible—the columns are linearly independent and we can see 2 pivots, so two of the conditions are clearly met.

(c) This is invertible so long as neither a nor b are zero, for the same reason (we need them to be pivots).

(d) This is invertible so long as the columns aren't linearly dependent. Since there's only two of them, this would only happen if one was a multiple of the other. That is, $b = sa$, $d = sc$ for some s not zero. We could write that as $\frac{b}{a} = s = \frac{d}{c}$ (so long as a and c aren't zero), which we can see implies that $ad = bc$ is the condition for the columns being linearly dependent (this works even if a or c is zero—we could have derived it another way in that case). Rearranging, we could have said the condition is $ad - bc = 0$. Thus, the matrix *is* invertible precisely when $ad - bc \neq 0$.

(e) Here, the columns are again linearly dependent, although it's a bit harder to see—the third is the sum of the first two. This might suggest that the condition for a 3×3 is a little more complicated than the 2×2 condition from the previous part, but we can get a condition.

(f) This is invertible so long as a, e, and i are all nonzero—we want them to be pivots.

The Definition: We define the *determinant* of a matrix in general to be

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

In this definition, we're moving along the first row, taking (-1) to be one power higher than the column we're in (this means take a positive value for odd columns and a negative one for even columns), multiplying by the entry we find, then taking the determinant of the smaller matrix obtained by ignoring the top row and the column we're in. This involves the determinant of smaller matrices, so if we drill down enough layers, we'll get back to 2×2 matrices, where we can just use the formula $ad - bc$ from earlier.

1 Find the determinants for each of these matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

(a) This is a 2×2, so we can just use the formula, obtaining $6 - 3(2) = 0$.

(b) Here $6 - 0(2) = 6$.

(c) More generally, $ad - 0b = ad$.

(d) We get $(-1)^{1+1} 1 \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 6 - (-4) = 10$.

(e) This is $1 \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 4 - (-1)(1) = 5$.

(f) More generally, $a \begin{vmatrix} e & f \\ 0 & i \end{vmatrix} - b \begin{vmatrix} 0 & f \\ 0 & i \end{vmatrix} + c \begin{vmatrix} 0 & e \\ 0 & 0 \end{vmatrix} = aei$. The same argument shows that the determinant of *any* triangular matrix can be found by multiplying the diagonal entries.

Cofactor Expansion: We can actually expand along *any* row or column. In that case, the (-1) has exponent $(-1)^{i+j}$ (where i marks the row and j the column we're in), the matrix entry is a_{ij}, and the subdeterminant is $\det(A_{ij})$. The goal here is to find the simplest row or column to move along to minimize the amount of computation. Mostly, this means finding the row or column with the most zeros, and the "nicest" (e.g., smallest) nonzero entries.

2 Try to compute the determinants of the following matrices by computing as few subdeterminants as possible.

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -2 & 0 & 3 \end{bmatrix}$ (min is 1 subdeterminant)

(b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$ (min is 2 subdeterminants)

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (min is 3 subdeterminants if you use a result from problem 1, and 4 otherwise)

(d) $\begin{bmatrix} 2 & -1 & 0 & 3 \\ 7 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 1 \end{bmatrix}$ (min is 2 subdeterminants)

For many of these, there are several ways to go which are equivalently easy, these are just some of them. When doing this kind of thing, just do what you find easiest.

(a) Going down the second column, we get $(-1)^{1+2}(1)\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} = -4$

(b) Going across the third row, we get $\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 4 - (-3) = 7$.

(c) Going down the first column, we get $\begin{vmatrix} 1 & 3 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 4 & 8 \\ 1 & 3 & 9 \\ 0 & 1 & 4 \end{vmatrix} = 1 - \left(2\begin{vmatrix} 3 & 9 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 4 & 8 \\ 1 & 4 \end{vmatrix}\right) = 1 - (6 - 8) = 3$. We went down the first column in the 3×3.

(d) Going down the third column, we get $(-1)^{2+3}\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 0 \\ -3 & -2 & 1 \end{vmatrix} = (-1)\left((-1)^{2+1}\begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix}\right) = 7$. We went across the second row in the 3×3.