Math 4A
Key 5.1
Lecture 11/20/17

Eigenvalues

The Punch Line: If we have a linear transformation from an n-dimensional vector space to itself, we can choose a basis that makes the matrix of the linear transformation especially simple—characterized by just n constants.

Warm-Up: Are the following matrices invertible? If not, what is the dimension of their null space?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(a) Invertible—its columns are clearly linearly independent.

(b) Invertible—again, the columns are linearly independent.

(c) Not invertible—the columns are the same. The null space has dimension 1—it is Span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

(d) Invertible—the determinant is $8 \neq 0$.

(e) Not invertible—the determinant is 0. The null space has dimension 1: we can see that there will be two pivots in the REF of the matrix, so it will have rank 2, hence by the Rank-Nullity Theorem ($\operatorname{rank} A + \dim \operatorname{Nul} A = n$ for an $m \times n$ matrix), $\dim \operatorname{Nul} A = 1$.

(f) Not invertible—the determinant is 0. The null space will have dimension 1 here, as it consists of all vectors with 0 as their second component (check this). It might be a good idea to keep this matrix in mind as we start exploring how to find the eigenvalues of a matrix—this one doesn't have "enough" eigenvectors (we'll see what this means later).

Eigenvalues and Eigenvectors: If V is an n-dimensional vector space and T is a linear transformation from V back into itself, and we find a (nonzero) vector $\vec{v} \in V$ and $\lambda \in \mathbb{R}$ that make the equation $T(\vec{v}) = \lambda\vec{v}$, we call λ and *eigenvalue* and $\vec{v}$ an *eigenvector* for T. The eigenvectors of the linear transformation are vectors whose direction does not change when you apply the transformation (except possibly reversing if $\lambda < 0$). The eigenvalues of the linear transformation are the different "scaling factors" that the transformation uses (as well as containing information about whether it reverses direction of certain vectors).

1 Are these vectors eigenvectors of the given linear transformation? If so, what are their eigenvalues?

(a) $\vec{v} = t^2 \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(b) $\vec{v} = t \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(c) $\vec{v} = 1 \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(d) $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\vec{x}$

(e) $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\vec{x}$

(f) $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix}1 & -5/3\\0 & -3/2\end{bmatrix}\vec{x}$

(g) $\vec{v} = \begin{bmatrix}0\\0\end{bmatrix}$ with $T(\vec{x}) = \vec{0}$

(a) Yes, with eigenvalue 2: $T(t^2) = t(2t) = 2t^2$.

(b) Yes, with eigenvalue 1: $T(t) = t(1) = t$.

(c) Yes, with eigenvalue 0: $T(1) = t(0) = 0$.

(d) No, because $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$.

(e) No, because $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$.

(f) Yes, because $\begin{bmatrix}1 & -5/3\\0 & -3/2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2 - \frac{5}{3}(3)\\ -\frac{3}{2}(3)\end{bmatrix} = \begin{bmatrix}-3\\ -\frac{9}{2}\end{bmatrix} = -\frac{3}{2}\begin{bmatrix}2\\3\end{bmatrix}$. (Sorry for the numbers on this one, but I wanted you to see that eigenvectors can be a little messy sometimes.)

(g) No, because in our definition we insist that eigenvectors be nonzero. In fact, $\vec{0}$ is the *only* vector that isn't an eigenvector of this T: all the other vectors are eigenvectors with eigenvalue 0. There's a good reason for excluding $\vec{0}$ as an eigenvector—we want the set of eigenvalues that a matrix has to be a description of what it does, and $\vec{0}$ would be an eigenvector for *every* eigenvalue.

Eigenspaces: The set of eigenvectors for eigenvalue λ of a given linear transformation is almost a subspace—all it's missing is the zero vector, which we may as well add in (after all, it also satisfies the equation $T(\vec{v}) = \lambda\vec{v}$). This means we can find a subspace corresponding to each eigenvalue of the linear transformation—we call it the *eigenspace for eigenvalue* λ, and denote it E_λ.

This is the null space of a linear transformation which is a slight modification of the original: if our transformation had matrix A, then E_λ is the null space of $A - \lambda I_n$. This is because if $(A - \lambda I_n)\vec{v} = 0$, then $A\vec{v} - \lambda I_n\vec{v} = A\vec{v} - \lambda\vec{v} = \vec{0}$, or $A\vec{v} = \lambda\vec{v}$. This means that if we know λ is an eigenvalue of the transformation, we can find its eigenspace by using techniques we already know for describing null spaces! We can also prove some number $\mu \in \mathbb{R}$ is *not* an eigenvalue by showing that its "eigenspace" (the null space of $A - \mu I_n$) is just $\left\{\vec{0}\right\}$, which isn't an eigenvector.

2 Determine if λ is an eigenvalue for the (transformation given by the) matrix A by computing E_λ:

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\lambda = 1$

(b) $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $\lambda = 2$

(c) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$

(d) A is an invertible matrix, $\lambda = 0$

(a) We look at $A - 1I_2 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$. The REF of this matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so there is only the trivial solution to the homogeneous equation for it, so the space $E_1 = \left\{\vec{0}\right\}$, so 1 is not an eigenvalue.

(b) We look at $A - 2I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$. This has REF $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, hence the homogeneous equation has parametric solution

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This means the null space of $A - 2I_2$ is Span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, so any vector with both components equal and nonzero is an eigenvector of A with eigenvalue 2.

(c) We look at $A - I_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. The REF of this matrix is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the null space is Span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$. Thus, $E_1 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(d) The eigenspace E_0 is the null space of $A - 0I_n = A$. If A is invertible, though, this means that its null space consists only of the zero vector. This shows that 0 is not an eigenvalue of any invertible matrix (or any invertible linear transformation in general)!

Under the Hood: Eigenspaces are a very interesting and important class of *invariant subspaces*—subspaces that are preserved by the transformation, in that any vector in the subspace will be mapped to another vector in the same subspace. The action of the transformation is very simple in the eigenspaces, so this is a huge win—if we can break any vector up into pieces that are all in eigenspaces, we can describe what happens to it just by seeing how each of those pieces gets scaled by the appropriate eigenvalue.

As it turns out, eigenspaces tell most of the story of invariant subspaces. There are only two other kinds in $\mathbb{R}^n$: spaces where the transformation looks like a rotation rather than a scaling, and spaces where the transformation "eventually" works like a scaling (after you apply it enough times). In this course, though, we'll just be focused on eigenspaces as they're presented here (ask me if you want to know more, or take Math 108).