Diagonalization

The Punch Line: Eigenvalues and -vectors can be used to factor a matrix in a way that makes computation easier.

Warm-Up What are the eigenvalues of these matrices? What are their eigenspaces?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

(a) This has eigenvalues $\lambda = 1, 2, 3$, with eigenspaces Span $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, respectively.

(b) This has eigenvalue $\lambda = 1, 2, 3$, with eigenspaces Span $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 9/2 \\ 3 \\ 1 \end{bmatrix}\right\}$, respectively.

(c) This has eigenvalues $\lambda = -1, 1$, with eigenspaces Span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, respectively.

(d) The characteristic equation here is

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 7\lambda = 0.$$

Thus, the eigenvalues are $\lambda = 0, 7$, with eigenspaces Span $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$, respectively.

(e) This has only the eigenvalue $\lambda = 3$, with eigenspace Span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

(f) We find the characteristic equation

$$\begin{vmatrix} -1-\lambda & -2 & 1 \\ -2 & 2-\lambda & -2 \\ 1 & -2 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -2 & -1-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ -2 & -1-\lambda \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 2-\lambda & -2 \end{vmatrix} = 16 + 12\lambda - \lambda^3 = 0.$$

We can factor this polynomial as $(4-\lambda)(2+\lambda)^2 = 0$, so our eigenvalues are $\lambda = -2, 4$. Checking their eigenspaces yields Span $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$, Span $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$, respectively.

Diagonalizing: If the matrix A has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ corresponding to them, then we write $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ for the matrix whose columns are the eigenvectors and D for the matrix with the eigenvalues down the diagonal and zeroes elsewhere. Then

$$AP = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD.$$

If the eigenvectors are linearly independent, then P is invertible, and $A = PDP^{-1}$.

1 Are these matrices diagonalizable? If so, what are P and D?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

(a) Yes, it is already diagonal! Here $P = I_3$ and D is the original matrix.

(b) Yes—we've found the eigenvalues and vectors in the warm-up, so $P = \begin{bmatrix} 1 & 2 & 9/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) Yes—$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Of course, we could also have chosen $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$—the order doesn't matter so long as the same order is used for the eigenvectors and -values.

(d) Yes again. Here $P = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$. Note that P is invertible, while D and our original matrix are not. Diagonalizability is a different condition from invertibility—non-invertible matrices like this one can be diagonalizable, and invertible matrices can be non-diagonalizable!

(e) This matrix is not diagonalizable—there is only one eigenvector, so we can't make an invertible matrix P out of eigenvectors! Note that this matrix is invertible—the inverse is $\frac{1}{9}\begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$—which shows off a sort of essential form of non-diagonalizable matrices (ask me if you're curious about this, there's some interesting stuff going on, but it's outside the scope of this course).

(f) This is diagonalizable! We have $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Again, we could have chosen P and D differently—in this case, it's worth noting that we could have chosen $P = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & -2 \\ 1 & 5 & 1 \end{bmatrix}$ with the same D, because we can take *any* two linearly independent eigenvectors from the two-dimensional eigenspace E_{-2}.

Using the Diagonalization: If we have written $A = PDP^{-1}$ with D a diagonal matrix, then we can easily compute the kth power of A as $A^k = PD^kP^{-1}$ (adjacent P and P^{-1} matrices will cancel, putting all of the D matrices together and just leaving the ones on the end).

2 The Fibonacci numbers are a *very* famous sequence of numbers. The first one is $F_1 = 0$, the second is $F_2 = 1$, and from then on out, each number is the sum of the previous two $F_n = F_{n-1} + F_{n-2}$ (this is sometimes used as a simple model for population growth—although it assumes immortality). Since it's annoying to compute F_n if n is very large (we'd have to do a lot of backtracking to get to known values), it would be nice to have a closed form for F_n. We can derive one with the linear algebra we already know!

(a) Since the equation defining F_n in terms of F_{n-1} and F_{n-2} is linear, we can use a matrix equation to represent the situation. In particular, we want a matrix A such that

$$A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

so that we can keep applying A to get further along in the sequence. What is this A?

(b) Since $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = A^2\begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix} = \cdots$, we can find F_n by computing $A^{n-2}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = A^{n-2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (we only need to advance by $n-2$ steps, because the top entry starts at 2). It's easier to raise matrices to powers after we diagonalize them, so find an invertible P and diagonal D so that $A = PDP^{-1}$ (the numbers are a little gross, so don't be alarmed).

(a) We want $A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n-1} + F_{n-2} \\ F_{n-1} \end{bmatrix}$, so we can find the matrix of the linear transformation

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0.$$

Using the quadratic formula on this, we see that the eigenvalues are $\lambda = \frac{1 \pm \sqrt{5}}{2}$ (the positive root is the Golden Ratio φ, while the negative root is sometimes denoted $\overline{\varphi}$). Looking at $A - \varphi I_2 = \begin{bmatrix} -\frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix}$, we see it has eigenvector $\begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix}$. Similarly, an eigenvector for $\overline{\varphi}$ is $\begin{bmatrix} -\sqrt{5} \\ 2 \end{bmatrix}$. Thus, we can write

$$A = \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\frac{1}{20}\begin{bmatrix} 2 & \sqrt{5} \\ -2 & \sqrt{5} \end{bmatrix}.$$

2 cont.

(c) Since $A^k = PD^kP^{-1}$, we can write out F_n as the first component of $PD^{n-2}P^{-1}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = PD^{n-2}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (if we wanted to be clever, we could write this as

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} PD^{n-2}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

as the row vector picks out the first component). Use this to write down a formula for F_n (don't worry about multiplying out powers of any terms involving square roots, just leave them as whatever they are)! Nifty!!!

(c) Since $P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/10 \\ -1/10 \end{bmatrix}$ and $D^{n-2} = \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \end{bmatrix}$, we get

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \end{bmatrix}\begin{bmatrix} 1/10 \\ -1/10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5}}{10}\left(\varphi^{n-2} + \overline{\varphi}^{n-2}\right) \\ \frac{2}{10}\left(\varphi^{n-2} - \overline{\varphi}^{n-2}\right) \end{bmatrix} = \frac{\varphi^{n-2} + \overline{\varphi}^{n-2}}{2\sqrt{5}}.$$

Under the Hood: The right way to think about the matrices P and P^{-1} is as change-of-coordinates matrices to an *eigenbasis*—then the requirement for diagonalizability is that the eigenvectors of A form a basis for the space they're in. Essentially, what we're doing is choosing a clever basis so that A looks like a diagonal matrix in that basis.