Math 4A
Worksheet 6.3
Lecture 12/04/17

Orthogonal Projection

The Punch Line: Inner products make it quite easy to compute the component of vectors that lie in interesting subspaces—in particular, components in the direction of any other vector.

Warm-Up What is the closest vector on the x-axis to the following vectors?

(a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \end{bmatrix}$

What is the closest point on the y-axis to these vectors? On the xy-plane?

(d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 9 \\ 1 \\ 2 \end{bmatrix}$ (f) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Orthogonal Projection: If we have some vector $\vec{u}$ that we're interested in, we can compute the *orthogonal projection* of any other vector $\vec{v}$ onto $\vec{u}$ as $\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. That is, the ratio of the inner product of $\vec{v}$ and $\vec{u}$ to the inner product of $\vec{u}$ with itself is the coefficient on $\vec{u}$ giving the closest vector in Span $\{\vec{u}\}$ to $\vec{v}$. This coefficient can be thought of as "the amount of $\vec{v}$ in the direction of $\vec{u}$", and the projection (which is a vector) as "the component of $\vec{v}$ in the direction of $\vec{u}$."

1 Compute the projection of $\vec{v}$ onto $\vec{u}$.

(a) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$

Projection Onto Subspaces: If W is a subspace of $\mathbb{R}^n$, we can compute the projection of a vector onto W. This is found by taking all and only the component of a vector which lie in W, which is most easily done if we have an orthogonal (or orthonormal) basis for W. Then we can simply compute the relevant inner products to project onto each basis vector, then add up all the results. (Note that this won't work if the basis isn't orthogonal.)

2 Project the vector $\vec{v}$ onto the subspace spanned by the given vectors.

(a) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Under the Hood: Why are orthogonal bases so much easier to project onto (we don't even have a good way to project onto the span of non-orthogonal vectors other than finding an orthogonal basis for that same subspace)? Heuristically, each vector in an orthogonal set is giving "independent information" about a vector in their span. Travelling in the direction of one of them doesn't move at all in the direction of the others, while for non-orthogonal vectors, increasing in one direction also moves in some of the others, and it's hard to separate the effects.

So, a basis gives enough information to describe any vector (it spans the space) and doesn't have redundant information (it's linearly independent), while an *orthogonal* basis also has the property that pieces of that description don't interfere with each other. An orthonormal basis is even nicer, in that the information requires less processing to get information about lengths—the coefficient on each component is the length in that direction (in other bases, the length of the basis vector changes this).