Math 4A
Worksheet 6.4
Lecture 12/06/17

The Gram-Schmidt Process

The Punch Line: We can turn any basis into an orthonormal basis using a (relatively) simple procedure.

Warm-Up For what choices of the variables are these bases orthogonal? Can they be made orthonormal by choosing variables correctly?

(a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}4\\y\end{bmatrix}, \begin{bmatrix}x\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}, \begin{bmatrix}x\\y\\0\end{bmatrix}, \begin{bmatrix}-x\\0\\z\end{bmatrix}\right\}$

The Gram-Schmidt Process: Suppose we know $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ is a basis for some subspace W we are interested in. We can make an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for the same subspace by repeatedly stripping away the parts of vectors that are not orthogonal to the previous ones.

In particular, we set $\vec{v}_1 = \vec{w}_1$ (there aren't previous vectors that it could be nonorthogonal to). Then we set $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$ (we take off any part of $\vec{w}_2$ that's in the direction $\vec{v}_1$ with a projection). Similarly, we set $\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$ (we have to remove parts in the first *two* directions now). In general, we set

$$\vec{v}_k = \vec{w}_k - \frac{\vec{w}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{w}_k \cdot v_{k-1}}{\vec{v}_{k-1} \cdot v_{k-1}}\vec{v}_{k-1}$$

(subtracting off the projection onto all previous vectors in the basis we are constructing).

1 Apply the Gram-Schmidt Process to the following (ordered) bases:

(a) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-3\end{bmatrix}, \begin{bmatrix}17\\-3\\1\end{bmatrix}\right\}$

Orthonormal Bases: After applying the Gram-Schmidt Process, it's easy to get an orthonormal basis—just rescale the results. It's important to note that the rescaling can be done right after subtracting off the projections onto the previous vectors, but shouldn't be done before doing so, as subtracting vectors changes lengths (it won't harm the process, but you won't get unit vectors out of it).

2	Find the orthonormal bases from the results of Problem 1.