(1) FIND THE EIGENVALUES & CORRESPONDING EIGENVECTORS.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

#1 $\det \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = 0$

$$(3-\lambda)(2-\lambda) - 2 = 0$$

$$\lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4 =$$

$$= (\lambda - 1)(\lambda - 4) = 0$$

$\lambda_1 = 1$ $\quad \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} 3x + y = x \\ 2x + 2y = y \end{cases} \iff \begin{cases} 2x + y = 0 \\ 2x + y = 0 \end{cases}$$

$$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\lambda_2 = 4$ CLEARLY $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

CHECKING THE ANSWER

1. $\det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = 6 - 2 = 4 = \lambda_1 \cdot \lambda_2 = 1 \cdot 4$

   $\text{TRACE} \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = 3 + 2 = 5 = \lambda_1 + \lambda_2 = 1 + 4$

2. $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

   $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

HENCE $\underset{A}{\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}} \underset{V}{\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}} = \underset{V}{\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}}$

$\underline{AV = VD}$ [redacted]

$\boxed{A = V D V^{-1}}$

A diagonalizable

So

$$A = V \sqrt{D} \sqrt{D} V^{-1} = V \sqrt{D} V^{-1} (\underbrace{V \sqrt{D} V^{-1}}_{\sqrt{A}})$$

$$\sqrt{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

#2 $\det(B - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = 0$

$$(2-\lambda)^2 + 1 = 0 \iff (2-\lambda)^2 = -1$$

$$\iff 2 - \lambda = \pm i \iff \boxed{\lambda = 2 \pm i}$$

$\lambda_1 = 2 + i$

$\lambda_2 = 2 - i$

CHECKING

$\det B = 5 = \lambda_1 \cdot \lambda_2$

$\text{TRACE}(B) = 4 = \lambda_1 + \lambda_2$.

V_1 $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (2+i) \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} 2x + y = (2+i)x \\ -x + 2y = (2+i)y \end{cases} \iff \begin{cases} y = ix \\ -x = iy \end{cases}$$

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\lambda_1 = 2+i \qquad v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\overline{\lambda_1} = \lambda_2$

$\overline{v_2} = v_1$

$\lambda_2 = 2-i \qquad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = (2+i) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (2-i) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{V} = \underbrace{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}}_{D}$$

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$ 5

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Leftrightarrow (1-\lambda)^2 = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 1$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x = x \\ x + y = y \end{cases} \Rightarrow x = 0 \quad y = 1$$

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

v_2 independent of v_1 does not exist

CINOT DIAGONALIZABLE